

On recursive properties of certain p -adic Whittaker functions

Fritz Hörmann

The Department of Mathematics and Statistics, McGill University, Montréal, Canada
E-mail: hoermann@math.mcgill.ca

Abstract

We investigate recursive properties of certain p -adic Whittaker functions (of which representation densities of quadratic forms are special values). The proven relations can be used to compute them explicitly in arbitrary dimensions, provided that enough information about the orbits under the orthogonal group acting on the representations is available. These relations have implications for the first and second special derivatives of the Euler product over all p of these Whittaker functions. These Euler products appear as the main part of the Fourier coefficients of Eisenstein series associated with the Weil representation. In case of signature $(m - 2, 2)$, we interpret these implications in terms of the theory of Borchers' products on orthogonal Shimura varieties. This gives some evidence for Kudla's conjectures in higher dimensions.

Keywords: Quadratic forms, Representation densities, Kudla's conjectures
2000 MSC: 14G40, 14G35, 11E45, 11E12

1. Introduction

Let L and M be \mathbb{Z} -lattices of dimension m and n equipped with non-degenerate quadratic forms. The purpose of this article is the investigation of the associated representation densities, defined for each prime p (in the simplest case) as the volume of

$$I(M, L)(\mathbb{Z}_p) = \{\alpha : M_{\mathbb{Z}_p} \rightarrow L_{\mathbb{Z}_p} \mid \alpha \text{ is an isometry}\},$$

w.r.t. some canonical volume form. They are determined by the number of elements in $I(M, L)(\mathbb{Z}/p^k\mathbb{Z})$ for sufficiently large k and are of considerable interest because

1. in the positive definite case, certain averages of the representation numbers $\# I(M, L^{(i)})(\mathbb{Z})$ over all classes $L^{(i)}$ in the genus of L are (up to a factor at ∞) equal to the Euler product over all representation densities by Siegel's formula.

2. in the indefinite case, the same kind of product gives the relative volume of certain *special cycles* on the locally symmetric orbifold associated with the lattice L .
3. the product may be understood as a (special value of a) Fourier coefficient of an Eisenstein series associated with the Weil representation [32]. This is related to 1., resp. 2., by the Siegel-Weil formula [33].

The construction of Eisenstein series defines a natural ‘interpolation’ of the individual factors in the Euler product as a function in $s \in \mathbb{C}$. These functions are p -adic Whittaker functions in the non-Archimedean case and confluent hypergeometric functions in the Archimedean case, respectively. All special values of the p -adic Whittaker functions at integral s are volumes of sets like $I(M, L \oplus H^s)(\mathbb{Z}_p)$, where H denotes an hyperbolic plane. Furthermore they are polynomials in p^{-s} and hence determined by these values.

If L has signature $(m-2, 2)$, the locally symmetric spaces associated with L , mentioned in 2. above, are in fact *Shimura varieties*. Kudla, motivated by the celebrated work of Gross and Zagier [8, 9], conjectured a general relation of a *special derivative* w.r.t. s of the Fourier coefficients of the same Eisenstein series to *heights* of the special cycles (see e.g. [16, 18]). Defining and working appropriately with these heights requires integral models of compactifications of the Shimura varieties in question. In the thesis of the author [11], a theory of those was developed and some of these conjectures could be partially verified in arbitrary dimensions. This required in particular a study of the recursive properties of the occurring p -adic Whittaker functions. This article is dedicated to a detailed discussion of those. It is organized as follows:

In section 3, we define the representation densities $\mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa)$ as the volume of $I(M, L)(\mathbb{Q}_p) \cap \kappa$, where κ is a coset in $(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}) \otimes M_{\mathbb{Z}_p}^*$, with respect to a certain canonical measure. They differ from the classical densities by a discriminant factor, but have much nicer recursive properties. All of these follow formally from the compatibility of the canonical measures with composition of maps (Theorem 5.2). We recover a classical recursive property due to Kitaoka (Corollaries 5.3, 5.4), as well as an easy orbit equation (Corollary 5.6) of the shape

$$\mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa) = \sum_{\text{orbits } \text{SO}'(L_{\mathbb{Z}_p})\alpha \text{ in } I(M, L)(\mathbb{Q}_p) \cap \kappa} \frac{\text{volume of } \text{SO}'(L_{\mathbb{Z}_p})}{\text{volume of } \text{SO}'(\alpha_{\mathbb{Z}_p}^\perp)}.$$

where SO' denotes the discriminant kernel. In section 4, we recall the Weil representation, the definition of the associated Eisenstein series and the relation of p -adic Whittaker functions to representation densities.

For the remaining part, we assume $p \neq 2$.

In section 5.8 ff., the volume of the discriminant kernel of the orthogonal group is ‘interpolated’ as a function $\lambda(L_{\mathbb{Z}_p}; s)$ by means of adding hyperbolic planes, too, which turns out to be a quite simple polynomial in p^{-s} (Theorem 8.1). We show that Kitaoka’s formula and the above orbit equation are true also for the p -adic Whittaker functions $\mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s)$ and the $\lambda_p(L_{\mathbb{Z}_p}; s)$. For this, we show

that SO' -orbits remain (ultimately) stable while adding hyperbolic planes. This reproves in an elementary way that the interpolated representation densities are polynomials in p^{-s} , too, for sufficiently large $s \in \mathbb{Z}_{\geq 0}$, and allows in principle to calculate these polynomials for arbitrary dimensions, provided one has enough information about the orbits. This is illustrated for $n = 1$ in section 9, where we recover a special case of Yang's explicit formula (Theorem 7.2).

Section 7 is dedicated to the case $\dim(M) = 1$. In this case the p -adic Whittaker function, as a polynomial in p^{-s} , may be computed by means of counting *all* $I(M, L)(\mathbb{Z}/p^k\mathbb{Z})$ up to some specified k . This yields a relation to zeta functions of the lattice, too — see Lemma 7.3. Furthermore, there exists a nice explicit formula due to Yang in this case (Theorem 7.2). A similar formula is actually proven for $p = 2$ in Yang's paper [35], too.

The development of the orbit equation, however, was motivated by the following. Assume $L_{\mathbb{Z}}$ is a global lattice of signature $(m - 2, 2)$. Consider the locally symmetric space associated with $L_{\mathbb{Z}}$, the orbifold

$$[SO(L_{\mathbb{Q}}) \backslash \mathbb{D}_{\mathbf{O}} \times (SO(L_{\mathbb{A}^{(\infty)}}) / SO'(L_{\widehat{\mathbb{Z}}}))],$$

where $\mathbb{D}_{\mathbf{O}}$ is the associated (Hermitian) symmetric space and $SO'(L_{\widehat{\mathbb{Z}}})$ is the discriminant kernel, a compact open subgroup of $SO(L_{\mathbb{A}^{(\infty)}})$. It is a Shimura variety in this case, having a canonical model $M^{(SO'(L_{\widehat{\mathbb{Z}}})\mathbf{O})}$ over \mathbb{Q} ($m \geq 3$). On it, we have the *special cycle* $Z(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa)$, defined analytically as (see 11.2 for details)

$$\sum_{SO'(L_{\widehat{\mathbb{Z}}})\alpha \subset I(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa} \left[SO(\alpha_{\mathbb{Q}}^{\perp}) \backslash \mathbb{D}_{\mathbf{O}(\alpha^{\perp})} \times (SO(\alpha_{\mathbb{A}^{(\infty)}}^{\perp}) / SO'(\alpha_{\widehat{\mathbb{Z}}}^{\perp})) \right],$$

i.e. as a sum of sub-Shimura varieties associated with certain lattices $\alpha_{\widehat{\mathbb{Z}}}^{\perp}$ (2.5). The volume of the Shimura variety $M^{(SO'(L_{\widehat{\mathbb{Z}}})\mathbf{O})}(\mathbb{C})$ (w.r.t. a specific automorphic volume form) is given roughly by the product over all ν of $\lambda_{\nu}^{-1}(L_{\mathbb{Z}_p}; 0)$ (with an appropriate factors for $\nu = \infty$). Written in the product over all ν in the form 10.5:

$$\lambda^{-1}(L_{\mathbb{Z}}; s) \mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) = \sum_{SO'(L_{\widehat{\mathbb{Z}}})\alpha \subset I(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa} \lambda^{-1}(\alpha_{\widehat{\mathbb{Z}}}^{\perp}; s),$$

the value at $s = 0$ expresses just the decomposition of the special cycle into sub-Shimura varieties (additivity of volume). It is therefore tempting to believe, that its derivative should express the equality of the *height* of the special cycles as the sum of heights of its constituents. This however can *not* be true in general because already for $n = 1$ the μ in this equation differs from the (holomorphic part) of the Fourier coefficient of the Eisenstein series by a factor of $|2d(M_{\mathbb{Z}})|^{-\frac{s}{2}}$ which comes from the Archimedean place (cf. Theorem 4.7). After incorporating a $|2d(M_{\mathbb{Z}})|_p^{\frac{s}{2}}$ into μ_p , and slightly modifying λ_p , it remains true roughly at those p , where there is only one orbit (10.5). In the simple case of signature $(1, 2)$, Witt rank 1 (modular curve) the correction in more ramified cases can be explained

by the existence of another, more deep local equation of this shape, related to the theta correspondence. This will be investigated briefly in section 12. Nevertheless, the orbit equation is already in this form technically very useful in the remaining cases, if one takes its derivative up to rational multiples of $\log(p)$, roughly for all p where there is more than 1 orbit.

The application to Kudla's program of this and an interpretation of the value and derivative of the orbit equation at $s = 0$ (case $n = 1$) in terms of Borchers' theory [2] is illustrated in section 11. It was the main achievement in the thesis of the author [11] and will be published in detail in a forthcoming paper [10]. As motivation, we recommend the reader to read this section first, consulting only necessary definitions and statements from the foregoing text.

2. Notation and basic definitions

2.1. Let R be a p.i.d., L be an R -lattice, i.e. a free R -module of finite rank. We use the following notation:

$$\begin{aligned} \mathrm{Sym}^2(L^*) &= \{\text{quadratic forms on } L\} &= ((L \otimes L)^s)^* \\ \mathrm{Sym}^2(L)^* &= \{\text{symm. bilinear forms on } L\} &= (L^* \otimes L^*)^s \\ \mathrm{Sym}^2(L) &= \{\text{quadratic forms on } L^*\} &= ((L^* \otimes L^*)^s)^* \\ \mathrm{Sym}^2(L^*)^* &= \{\text{symm. bilinear forms on } L^*\} &= (L \otimes L)^s, \end{aligned}$$

here $(\cdots)^s$ denotes symmetric elements, i.e. invariants under the automorphism switching factors. Usually a non-degenerate quadratic form in $\mathrm{Sym}^2(L^*)$ will be fixed and denoted by Q_L . Its associated bilinear form $v, w \mapsto Q_L(v + w) - Q_L(v) - Q_L(w)$ is denoted by $\langle \cdot, \cdot \rangle_Q$, and its associated morphism $L \rightarrow L^*$ by γ_Q .

We denote the discriminant of Q_L , i.e. the determinant of γ_Q w.r.t. some basis of L by $d(L)$. It is determined up to $(R^*)^2$. We denote by $\langle \varepsilon_1, \dots, \varepsilon_m \rangle$ the lattice R^m with quadratic form $x \mapsto \sum_{i=1}^m \varepsilon_i x_i^2$ (R is always understood from the context). It has discriminant $2^m \prod_{i=1}^m \varepsilon_i$.

2.2. In this article, we work with the following natural (up to a choice of $i \in \mathbb{C}$) characters on $R = \cdots$:

$$\mathbb{R}: \chi_\infty(x) := e^{2\pi i x},$$

$$\mathbb{Q}_p: \chi_p(x) := e^{-2\pi i [x]}, \text{ where } [x] = \sum_{i < 0} x_i p^{-i} \text{ is the principal part,} \\ \text{(it has level/conductor 1),}$$

$$\mathbb{A}^S: \chi = \prod_{\nu \notin S} \chi_\nu.$$

The corresponding self-dual additive Haar measures are the Lebesgue measure on \mathbb{R} , the standard measures on \mathbb{Q}_p , giving \mathbb{Z}_p the volume 1, and their product, respectively.

2.3. Let R be one of the rings of 2.2. Let X be an algebraic variety over R and $\tilde{\mu}$ an algebraic volume form on X . As is explained in [34] (cf. also [29, §3.5]),

this defines a well-defined measure μ on $X(R)$, which depends on the choice of χ (resp. the additive Haar measure). For the special case of a lattice L of dimension r , and $\tilde{\mu} \in \Lambda^r L^*$, there is $\tilde{\mu}^* \in \Lambda^r L$ satisfying $\tilde{\mu}^* \tilde{\mu} = 1$. In this case, the measures μ and μ^* are dual to each other with respect to the bicharacter $v, v^* \mapsto \chi(v^* v)$, i.e. for

$$\begin{aligned} F_\Psi(w^*) &= \int_L \Psi(w) \chi(w^* w) \mu(w) & \Psi \in S(L) \\ F_\Psi(w) &= \int_{L^*} \Psi(w^*) \chi(w^* w) \mu^*(w^*) & \Psi \in S(L^*) \end{aligned}$$

(where $S(\dots)$ denotes space of Schwartz-Bruhat functions), we have $F_{F_\Psi}(w^*) = \Psi(-w^*)$.

2.4. Let L be an R -lattice with non-degenerate quadratic form Q_L . Then there is a canonical (translation invariant) measure μ_L with $\mu_L^* = \mu_L$ under the identification $\gamma_Q : L \xrightarrow{\sim} L^*$. Let e_1, \dots, e_m be a basis of L , e_1^*, \dots, e_m^* the dual basis and $\tilde{\mu} = e_1^* \wedge \dots \wedge e_m^*$. Let A be the matrix of \langle, \rangle_Q in this basis. The measure μ_L is then given by

$$\mu_L = |A|^{1/2} \mu,$$

where $|A|$ is the modulus of the determinant. We call it the **canonical measure** on L with respect to Q_L .

Let M be another R -lattice, equipped with a non-degenerate quadratic form Q_M .

Choose a basis f_1, \dots, f_n of M , too, and denote $\tilde{\mu} := \bigwedge_{i,j} e_i^* \otimes f_j^* \in \bigwedge^{nm} L^* \otimes M^*$.

We call $\mu_{L,M} = |A|^{n/2} |B|^{m/2} \mu$ the **canonical measure** on $L \otimes M$, where A are B the matrices of the associated bilinear forms, $m = \dim(L)$ and $n = \dim(M)$. $\{e_i \otimes e_j\}_{i \leq j}$ is a basis of $\text{Sym}^2(L)$. We denote the corresponding dual basis by $\{(e_i \otimes e_j)^*\}_{i \leq j}$. Let $\tilde{\mu} = \bigwedge_{i < j} (e_i \otimes e_j)^* \in \bigwedge^{\frac{m(m+1)}{2}} \text{Sym}^2(L)^*$. In this case we call $\mu_L = |A|^{\frac{m+1}{2}} \mu$ the **canonical measure** on $\text{Sym}^2(L)$.

Let $\tilde{\mu} = \bigwedge_i (e_i \otimes e_i)^* \wedge \bigwedge_{i < j} (e_i \otimes e_j + e_j \otimes e_i)^*$. In this case, we call $\mu_L = |A|^{\frac{m+1}{2}} \mu$ the **canonical measure** on $(L \otimes L)^s$.

Similarly, we get a **canonical measure** $\mu_L = |A|^{\frac{n-1}{2}} \mu$ on $\Lambda^2 L$.

According to these definitions, the measures μ_L on $\text{Sym}^2(L)$ and μ_L on $(L^* \otimes L^*)^s = \text{Sym}^2(L)^*$ are dual. However, the symmetrization map $\text{Sym}^2(L^*) \xrightarrow{\sim} \text{Sym}^2(L)^*$ sends the canonical measure μ_L of the left hand side to the $|2|^{-m}$ -multiple of μ_L on the right hand side.

2.5. Let R be a ring. Let L, M be R -lattices of rank m and n with quadratic forms Q_L and Q_M . Assume Q_L non-degenerate. For each R -algebra R' , we define

$$\text{I}(M, L)(R') := \{\alpha : M_{R'} \rightarrow L_{R'} \mid \alpha \text{ is an isometry}\}.$$

$\text{I}(M, L)$ is an affine algebraic variety, defined over R . If Q_M is degenerate, define in addition:

$$\text{I}^1(M, L)(R') := \{\alpha : M_{R'} \rightarrow L_{R'} \mid \alpha \text{ is an injective isometry}\}.$$

If $R = \mathbb{Q}_p$ or $\mathbb{A}^{(\infty)}$, for any compact open subgroup $K \subset \mathrm{SO}(L_R)$ and compact open subset $\kappa \subset L_R \otimes M_R^*$ there are finitely many K -orbits in $\mathrm{I}(M, L)(R) \cap \kappa$. We will write frequently

$$\sum_{K\alpha \subseteq \mathrm{I}(M, L)(R) \cap \kappa} \dots$$

meaning, that we sum over all those orbits and α is a respective representative. In the case $R = \mathbb{A}^{(\infty)}$, by $\alpha_{\mathbb{Z}}^{\perp}$ we understand a lattice which satisfies $\alpha_{\mathbb{Z}}^{\perp} \otimes \widehat{\mathbb{Z}} \cong \mathrm{im}(\alpha)^{\perp} \cap L_{\widehat{\mathbb{Z}}}$. It can be realized as $(\alpha')^{\perp} \cap L'_{\mathbb{Z}}$ where $L'_{\mathbb{Z}}$ is in the genus of $L_{\mathbb{Z}}$ satisfying $gL'_{\mathbb{Z}} = L_{\widehat{\mathbb{Z}}}$ for a $g \in \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ with $g\alpha' = \alpha$ for a $\alpha' \in \mathrm{I}(M, L)(\mathbb{Q})$. Such α' and g exist because of Hasse's principle and Witt's theorem, respectively. (In our global cases always $\mathrm{I}(M, L)(\mathbb{R}) \neq \emptyset$). However, only its genus is determined and all occurring formulas/objects will depend only on it.

Definition 2.6.

$$\Gamma_n(s) := \pi^{\frac{n(n-1)}{4}} \prod_{k=0}^{n-1} \Gamma(s - \frac{k}{2}).$$

$$\Gamma_{n,m}(s) := 2^n \frac{\pi^{\frac{n}{2}(s+m)}}{\Gamma_n(\frac{1}{2}(s+m))} = \prod_{k=m-n+1}^m 2 \frac{\pi^{\frac{1}{2}(s+k)}}{\Gamma(\frac{1}{2}(s+k))}.$$

Γ_n is the higher dimensional gamma function defined e.g. in [31].

3. Representation densities

3.1. Let $L_{\mathbb{Q}_p}, M_{\mathbb{Q}_p}$ be finite dimensional \mathbb{Q}_p -vector spaces with non-degenerate quadratic forms Q_L and Q_M , respectively and let $L_{\mathbb{Z}_p} \subset L_{\mathbb{Q}_p}, M_{\mathbb{Z}_p} \subset M_{\mathbb{Q}_p}$ be \mathbb{Z}_p -lattices.

The main object of study of this paper are the associated representation densities, i.e. the **volumes** of $\mathrm{I}(M, L)(\mathbb{Z}_p)$ or more generally the integral

$$\mu(L, M, \varphi) := \int_{\mathrm{I}(M, L)(\mathbb{Q}_p)} \varphi(x^*) \mu_{L, M}(x^*)$$

of a function $\varphi \in S(M^* \otimes L)$ (locally constant with compact support) with respect to a suitable measure $\mu_{L, M}$. The canonical measures in 2.4 induce a *canonical* measure $\mu_{L, M}$ on $\mathrm{I}(M, L)(\mathbb{Q}_p)$, too. In this section, we will first describe this measure (also for the real case) and then relate the volumes, respectively integrals to the classical definition of representation density.

3.2. Let R be a \mathbb{Q}_p or \mathbb{R} . Let L, M be vector spaces of dimension m, n . Assume $m \geq n \geq 0$. We identify $M^* \otimes L$ with $\mathrm{Hom}(M, L)$ in what follows. There is a fibration

$$\mathrm{I}(M^Q, L) \hookrightarrow M^* \otimes L \xrightarrow{\alpha \mapsto \alpha' Q_L} \mathrm{Sym}^2(M^*), \quad (1)$$

where $I(M^Q, L)$ is the preimage of $Q = Q_M$ and $\alpha^!Q_L$ denotes pullback of Q_L to M via α . As soon as we choose (translation invariant) measures $\mu_{1,2}$ on $M^* \otimes L$ and $\text{Sym}^2(M^*)$ respectively, this defines a measure $\frac{\mu_1}{\mu_2}$ on the fibers, restricted to the submersive set $(M_R^* \otimes L_R)^{reg}$ of the map $\alpha \mapsto \alpha^!Q_L$. This set coincides with the locus of maps with maximal rank n . This means in particular that the following integral formula holds true:

$$\int_{\text{Sym}^2(M_R^*)} \mu_2(Q) \int_{I(M^Q, L)(R)} \varphi(\alpha) \frac{\mu_1}{\mu_2}(\alpha) = \int_{M_R^* \otimes L_R} \varphi(\alpha) \mu_1(\alpha), \quad (2)$$

where φ is continuous with compact support on $(M_R^* \otimes L_R)^{reg}$.

Lemma 3.3 (Weil [33, §34]). *For $m \geq 2n + 1$ and R local, $\varphi \in S(M_R^* \otimes L_R)$ (space of Schwartz-Bruhat functions) is integrable with respect to the measure $\frac{\mu_1}{\mu_2}$, too.*

In the case $m \geq 2n + 1$, the integrals $\int_{I(M^Q, L)} \varphi(\alpha) \frac{\mu_1}{\mu_2}(\alpha)$ may be computed by Fourier analysis (cf. Theorem 7.2), via the following well-known theorem. It is also central for the connection between Eisenstein series (or Whittaker functions) and volumes. It is analogous to the connection between Gauss sums and representation numbers over finite fields.

Theorem 3.4 (Weil [33, Proposition 6]). *Let R be local and $m \geq 2n + 1$. Let $\varphi \in S(M_R^* \otimes L_R)$. The function*

$$\Psi(Q) = \int_{I(M^Q, L)} \varphi(\alpha) \frac{\mu_1}{\mu_2}(\alpha) \quad Q \in \text{Sym}^2(M_R^*)$$

has Fourier transform

$$\Psi'(\beta) = \int_{M^* \otimes L} \varphi(x^*) \chi((x^*)^!Q_L \cdot \beta) \mu_1(x^*) \quad \beta \in (M_R \otimes M_R)^s$$

with respect to a measure μ_2 on $\text{Sym}^2(M_R^)$. Here μ_1 , μ_2 and $\frac{\mu_1}{\mu_2}$ are connected via the fibration 3.2.*

Definition 3.5. *In particular, if Q_M is non-degenerate, and for any m, n , the canonical measures on $M_R^* \otimes L_R$ and $\text{Sym}^2(M_R^*)$, introduced in 2.4, define a **canonical measure** $\mu_{L, M}$ on the fibre $I(M, L)(R)$ (which is the fibre above $Q = Q_M$) by means of this fibration.*

We especially get a canonical and also invariant measure on every $\text{SO}(L_R)$, coming (up to a real factor) from an algebraic volume form. On the other hand, algebraic invariant volume forms on $\text{SO}(L_R)$ are canonically identified with $\Lambda^{\frac{m(m-1)}{2}} \text{Lie}(\text{SO}(L_R))^*$. Hence every invariant measure on $\text{SO}(L_R)$ is given by a translational invariant measure on $\text{Lie}(\text{SO}(L_R))^*$. We have the following

Lemma 3.6. *The associated measure on $\text{Lie}(\text{SO}(L_R))$ is the canonical measure (cf. 2.4) on $\Lambda^2 L_R$ under the natural identification $\text{Lie}(\text{SO}(L_R)) \cong \Lambda^2 L_R$ given by contraction with the bilinear form associated with Q_L .*

The relation to the classical definition of representation density (p -adic case) and classical sphere volumes (real case) is given by the following easy

Lemma 3.7.

1. For $n = m = 1$, $I(M, L)_R$ consists of two points, each of which has volume one.
2. For $R = \mathbb{R}$ and positive definite spaces $M_{\mathbb{R}}, L_{\mathbb{R}}$, we get

$$\text{vol}(I(M, L)(\mathbb{R})) = \prod_{k=m-n+1}^m 2 \frac{\pi^{k/2}}{\Gamma(k/2)}.$$

3. For $R = \mathbb{Q}_p$, choose lattices $L_{\mathbb{Z}_p}$ and $M_{\mathbb{Z}_p}$ and bases $\{f_i\}$ of $M_{\mathbb{Z}_p}$ and $\{e_i\}$ of $L_{\mathbb{Z}_p}$, respectively. Then we have:

$$\int_{I(M, L)(\mathbb{Q}_p)} \varphi(x^*) \mu_{L, M}(x^*) = D(M_{\mathbb{Z}_p}, L_{\mathbb{Z}_p}) \beta(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \varphi)$$

where

$$D(M_{\mathbb{Z}_p}, L_{\mathbb{Z}_p}) := |d(L_{\mathbb{Z}_p})|_p^{n/2} |d(M_{\mathbb{Z}_p})|_p^{(n-m+1)/2}$$

and

$$\beta(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \varphi) := \lim_{l \rightarrow \infty} p^{l(n(n+1)/2 - mn)} \sum_{\substack{\{\delta_i\} \subset p^{-r} L_{\mathbb{Z}_p} / p^l L_{\mathbb{Z}_p} \\ Q_L(\delta_i) \equiv Q_M(f_i) \pmod{p^l} \\ \langle \delta_i, \delta_j \rangle_L \equiv \langle f_i, f_j \rangle_M \pmod{p^l}}} \varphi\left(\sum_i f_i^* \otimes \delta_i\right)$$

and r is any integer with $\text{supp}(\varphi) \subseteq p^{-r} L_{\mathbb{Z}_p}$.

4. Fourier coefficients of Eisenstein series

We briefly recall the connection between representation densities and Eisenstein series associated with the Weil representation.

4.1. Let R be \mathbb{Q}_p , \mathbb{R} , or \mathbb{A} and χ the corresponding standard character (cf. 2.2). Let $L_{\mathbb{Q}}$ be a quadratic space of dimension m and $M_{\mathbb{Q}}$ be any finite dimensional space of dimension n . Let $\text{Sp}'(\mathfrak{M}_R)$ be $\text{Sp}(\mathfrak{M}_R)$ if m is even and $\text{Mp}(\mathfrak{M}_R)$ (metaplectic double cover) if m is odd. Here $\mathfrak{M} = M \oplus M^*$ with its natural symplectic form

$$\left\langle \begin{pmatrix} w_1 \\ w_1^* \end{pmatrix}, \begin{pmatrix} w_2 \\ w_2^* \end{pmatrix} \right\rangle \mapsto w_1^*(w_2) - w_2^*(w_1).$$

We denote:

$$g_l(\alpha) := \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{pmatrix} \quad (3)$$

$$u(\beta) := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \quad (4)$$

$$d(\gamma) := \begin{pmatrix} 0 & -{}^t\gamma^{-1} \\ \gamma & 0 \end{pmatrix} \quad (5)$$

for $\alpha \in \text{Aut}(M)$, $\beta \in \text{Hom}(M^*, M)$ and $\gamma \in \text{Iso}(M, M^*)$ and denote the image of g_l and u in $\text{Sp}(\mathfrak{M})$ by G_l and U respectively.

Recall [32, 33] the Weil representation $\omega_{L,M}$ of $\text{Sp}'(\mathfrak{M}_R) \times \text{SO}(L_R)$ on $S(M_R^* \otimes L_R)$ (space of Schwartz-Bruhat functions). It actually comes from the restriction of the Weil representation of $\text{Mp}(\mathfrak{M}_R \otimes L_R)$ on $S(M_R^* \otimes L_R)$. The above elements have lifts to Sp' which act via the formulæ:

$$\begin{aligned}\omega_{L,M}(g_l(\alpha))\varphi : \quad x^* &\mapsto \frac{\tilde{\Upsilon}(\gamma_0\alpha)}{\tilde{\Upsilon}(\gamma_0)} |\alpha|^{\frac{m}{2}} \varphi({}^t\alpha x^*) \\ \omega_{L,M}(u(\beta))\varphi : \quad x^* &\mapsto \chi((x^*)^! Q_L \cdot \beta) \varphi(x^*) \\ \omega_{L,M}(d(\gamma))\varphi : \quad x^* &\mapsto \tilde{\Upsilon}(\gamma) \int_{M \otimes L^*} \varphi({}^t\gamma x) \chi(-\langle x^*, x \rangle) \mu_{L,\gamma}(x)\end{aligned}$$

Here $\tilde{\Upsilon}(\gamma)$ is a certain eighth root of unity. It is equal to $\Upsilon(\gamma \otimes Q_L)$ if $n = 1$, where Υ is the Weil index, denoted by γ in [32, 33]. Warning: If m is odd, the lift of $g_l(\alpha)$ does not constitute a group homomorphism. $\mu_{L,\gamma}$ is the canonical measure on $L^* \otimes M$ determined by Q_L on L and γ on M .

We have obviously by the characterization of the Weil representation

$$\Psi'(\beta) = \tilde{\Upsilon}(\gamma_0)^{-1} (\omega_{L,M}(d(\gamma_0)u(\beta))\varphi)(0),$$

for the function Ψ' from Theorem 3.4. Here, we assume, that the measure μ_1 of that theorem has been chosen to be the canonical measure on $L \otimes M^*$ induced by γ_0 on M and Q_L on L .

This is the starting point for the connection of representation densities to Eisenstein series, as we will now briefly recall.

4.2. Let R be \mathbb{Q}_p , \mathbb{R} , or \mathbb{A} . Choose a maximal compact subgroup K'_R of $\text{Sp}'(R)$. We have the Iwasawa decomposition

$$\text{Sp}'(\mathfrak{M}_R) = P'K'_R.$$

where P' is the preimage of $P = UG_l$.

Let ξ' denote a character of $P'(\mathbb{A})$, which is trivial on $U(\mathbb{A})$ and on $P(\mathbb{Q})$, but nontrivial on the metaplectic kernel (if m is odd). If m is even, ξ comes from a character ξ of $\mathbb{A}^*/\mathbb{Q}^*$ lifted to G_l via $g_l(\alpha) \mapsto \xi(\det(\alpha))$.

Let $\text{I}_R(s, \xi'_R)$ be the (normalized) parabolically induced representation

$$\text{I}_R(s, \xi'_R) := \text{I}_{P'}^{\text{Sp}'(\mathfrak{M}, R)}(|\det|^s \xi'_R),$$

which is the space of smooth K_R -finite functions Ψ , satisfying

$$\Psi(pg) = \xi'_R(p) |\det(\alpha(p))|^{s + \frac{n+1}{2}} \Psi(g),$$

where $\alpha(p)$ is the G_l component in the Iwasawa decomposition of the projection of p to Sp .

In particular, for any R as above, the Weil representation defines the following $\mathrm{Sp}'(\mathfrak{M}_R)$ -equivariant operator:

$$\begin{aligned} \Phi : S(M_R^* \otimes L_R) &\rightarrow \mathrm{I}_R(\frac{m-n-1}{2}, \xi'_R) \\ \varphi &\mapsto \{g \mapsto (\omega_{L,M}(g)\varphi)(0)\} \end{aligned}$$

Here ξ' is determined by the character (up to sign, if m is odd) $\alpha \mapsto \tilde{\Upsilon}(\gamma_0\alpha)/\tilde{\Upsilon}(\gamma_0)$ for some γ_0 .

4.3. Let $\Psi(s_0) \in \mathrm{I}(s_0, \xi')$ be given (here we set $s_0 := \frac{m-n-1}{2}$). It can be extended uniquely to a ‘section’ parameterized by $s \in \mathbb{C}$, with the property that the restriction to K' is independent of s .

To any such ‘section’, there is an associated **Eisenstein series**. This association is a $\mathrm{Sp}'(\mathbb{A})$ -equivariant map

$$\begin{aligned} E : \mathrm{I}_{\mathbb{A}}(s, \xi') &\rightarrow \mathcal{A}(\mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}}) \backslash \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}})) \\ \Psi(s) &\mapsto \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}})} \Psi(s)(\gamma g), \end{aligned}$$

where \mathcal{A} is the space of automorphic functions. This series converges absolutely if $\Re(s) > \frac{n+1}{2}$ and possesses a meromorphic continuation in s to all of \mathbb{C} . Note that $\mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}})$ lifts canonically to $\mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}})$.

The Eisenstein series decomposes as follows:

$$E(\Psi, g; s) = \sum_{M^{*'} \subset M^*} E_{M^{*'}}(\Psi, g; s),$$

with

$$E_{M^{*'}}(\Psi, g; s) = \sum_{\beta \in (M' \otimes M')_{\mathbb{Q}}^s} \Psi(s)(d_{M'}(\gamma_0)u(\beta)g).$$

where $d_{M'}(\gamma_0)$ is embedded via $\mathrm{Sp}'(\mathfrak{M}') \hookrightarrow \mathrm{Sp}'(\mathfrak{M})$. This embedding, as well as the dual subspace $M' \subset M$, depend on the choice of a complement $(M^*)''$. One easily sees, however, that $E_{M^{*'}}(\Psi, g; s)$ does not depend on these choices, nor on the isomorphism $\gamma_0 : M'_{\mathbb{Q}} \rightarrow (M^*_{\mathbb{Q}})'$.

At $s = s_0$, with $m > 2n + 2$ (this assures convergence), we get:

$$E_{M^{*'}}(\Psi, g; s_0) = \sum_{\beta \in (M' \otimes M')_{\mathbb{Q}}^s} (\omega_{L,M'}(d(\gamma_0)u(\beta))\omega_{L,M}(g)\varphi)(0).$$

Using Theorem 3.4 and Poisson summation this yields:

$$E_{M^{*'}}(\Phi(\varphi), g; s_0) = \sum_{Q \in \mathrm{Sym}^2(M^{*'})_{\mathbb{Q}}} \int_{\mathrm{I}_{\mathbb{A}}(M'^Q, L)} (\omega_{L,M'}(g)\varphi)(x^*) \, dx^*.$$

Here we interpret $g\varphi$ by composition with the embedding $M^{*'} \otimes L \hookrightarrow M^* \otimes L$ as a function on $(M^{*'} \otimes L)_{\mathbb{A}}$. dx^* is the Tamagawa measure.

This is essentially the Fourier expansion of the Eisenstein series: A calculation shows that

$$\begin{aligned} E_Q(\Phi(\varphi), g; s_0) &= \int_{(M \otimes M)^* \setminus (M \otimes M)_{\mathbb{A}}^s} \sum_{M^{**'} \subset M^*} E_{M^{*'}}(\Phi(\varphi), u(\beta)g; s_0) \chi(\beta Q) d\beta \\ &= \sum_{M^{*'} \subset M^*} \int_{(M' \otimes M')_{\mathbb{A}}^s} (\omega_{L, M'}(d(\gamma_0)u(\beta)) \omega_{L, M}(g) \varphi)(0) \chi(\beta Q) d\beta \end{aligned}$$

It is convenient to make the following

Definition 4.4. *The Whittaker function is defined as*

$$W_{\nu, Q, M^{*'}}(\Psi, g) := \int_{(M' \otimes M')_{\mathbb{Q}_\nu}^s} \Psi(\omega_{L, M'}(d(\gamma_0)u(\beta)) \omega_{L, M}(g)) \chi(\beta Q) \mu_{\gamma_0}(\beta_\nu),$$

where we now chose the canonical measure μ_{γ_0} with respect to some fixed positive definite γ_0 on $M_{\mathbb{Q}}$ and hence $M'_{\mathbb{Q}}$ for convenience.

Hence we have

$$E_Q(\Psi, g; s) = \sum_{\{M^{*'} \subset M^* \mid Q \in \text{Sym}^2(M^{*'})\}} \prod_{\nu} W_{\nu, Q, M^{*'}}(\Psi_{\nu}(s), g_{\nu})$$

and again by Theorem 3.4:

$$W_{\nu, Q, M^{*'}}(\Phi(\varphi_{\nu})(s_0), g) = \tilde{\Upsilon}_{\nu}(\gamma_0) \int_{I_{\mathbb{Q}_\nu}(M'^Q, L)} (\omega_{L, M'}(g) \varphi_{\nu})(x_{\nu}^*) \mu_{L, \gamma_0}(x_{\nu}^*),$$

where μ_{L, γ_0} is the measure on $I_{\mathbb{Q}_\nu}(M'^Q, L)$ induced by the canonical ones on $\text{Sym}^2(M^*)$ and $M^* \otimes L$ with respect to γ_0 and Q_L via 3.2.

More generally, the Siegel-Weil formula equals the whole Eisenstein series associated with $\Phi(\varphi)$ to an integral of a theta function associated with φ (see e.g. [21, 22, 33]).

Assume, that γ_0 and a lattice $M_{\mathbb{Z}_p}$ are chosen such that γ_0 induces an isomorphism $M_{\mathbb{Z}_p} \rightarrow M_{\mathbb{Z}_p}^*$. We will now investigate the Whittaker integral as a function of s in the case $R = \mathbb{Q}_p$ to some extent.

Theorem 4.5. *Let $Q \in \text{Sym}^2(M^*)$ be non-degenerate (with associated bilinear form γ) and $r \in \mathbb{Z}_{\geq 0}$. Denote $s = s_0 + r$. Let $\varphi \in S(L_{\mathbb{Q}_p} \otimes M_{\mathbb{Q}_p}^*)$. For sufficiently large r , we have the equality:*

$$\begin{aligned} W_{Q, p}(\Phi(\varphi)(s), g) &= \tilde{\Upsilon}_p(\gamma_0) \int_{I_{\mathbb{Q}_p}(M^Q, L \perp H^r)} (\omega_{L \perp H^r, M}(g) \varphi^{(r)})(x^*) \mu_{\gamma_0, L \perp H^r}(x^*), \\ &= \tilde{\Upsilon}_p(\gamma_0) |\gamma|_p^s \mu_p(L, M_{\mathbb{Z}_p}^Q, \omega_{L, M}(g) \varphi; s - s_0) \end{aligned}$$

where $\varphi^{(r)}$ is φ tensored with the characteristic function $\chi^{(r)}$ of $H_{\mathbb{Z}_p}^r \otimes M_{\mathbb{Z}_p}^*$ (depends on the choice of $M_{\mathbb{Z}_p}$) and

$$\mu_p(L, M_{\mathbb{Z}_p}, \varphi; r) = \mu_p(L \perp H^r, M, \varphi^{(r)})$$

for $r \in \mathbb{Z}_{\geq 0}$. (Note, that the continuation of μ_p depends on the choice of the lattice $M_{\mathbb{Z}_p}$!)

Furthermore the left hand side is a polynomial in p^{-s} and therefore determined by the above values. Here $|\gamma|$ is computed with respect to the measure μ_{γ_0} on M .

Proof. The Weil representation on $S(L^{(r)} \otimes M^*)$ is the tensor product of the respective Weil representation on $S(L \otimes M^*)$ and $S(H^{(r)} \otimes M^*)$. We have

$$(\omega_{H^r, M}(u(\beta)g_l(\alpha)k)\chi^{(r)})(0) = |\alpha|^r,$$

where $k \in K$ and K is the maximal compact open subgroup associated with the lattice $\mathfrak{M}_{\mathbb{Z}_p}$. Hence multiplying $\Phi(\varphi)(s)$ by $\Phi(\chi^{(r)})$ or substituting s by $s + r$ has the same effect.

The assertion that the left hand side is a polynomial in p^{-s} follows from the arguments given in [30, p. 101]. We will later (cf. 5.11) prove directly that $\mu_p(\dots; s)$ is a polynomial in p^{-s} for sufficiently large s . \square

4.6. We now investigate the Whittaker integral *as a function of s* in the case $R = \mathbb{R}$ and for the function $\Psi_\infty := \Psi_{\infty, \frac{m}{2}}$, whose restriction to K' (maximal compact, see below) is $\det^{\frac{m}{2}}$. We again consider only the case that Q is non-degenerate on M . The maximal compact subgroup K of $\mathrm{Sp}(\mathfrak{M}_{\mathbb{R}})$ is defined as follows:

Let γ_0 be a symmetric and positive definite form on M . It defines an isomorphism:

$$\begin{aligned} M_{\mathbb{C}}^* &\rightarrow \mathfrak{M}_{\mathbb{R}} \\ w_1 + iw_2 &\mapsto w_1 - {}^t\gamma_0^{-1}w_2 = w_1 - \gamma_0^{-1}w_2 \end{aligned}$$

and a corresponding map

$$\begin{aligned} k : \mathrm{End}(M_{\mathbb{C}}^*) &\rightarrow \mathrm{End}(\mathfrak{M}_{\mathbb{R}}) \\ \alpha_1 + i\alpha_2 &\mapsto \begin{pmatrix} \gamma_0^{-1}\alpha_1\gamma_0 & -\gamma_0^{-1}\alpha_2 \\ \alpha_2\gamma_0 & \alpha_1 \end{pmatrix}. \end{aligned}$$

This identifies the unitary group of the Hermitian form given by γ_0 on $M_{\mathbb{C}}$ with the stabilizer of $d(\gamma_0)$ in $\mathrm{Sp}(\mathfrak{M}_{\mathbb{R}})$, which is a maximal compact subgroup, denoted K . Everything depends on the choice of γ_0 . We define K' to be the preimage of K in $\mathrm{Sp}'(\mathfrak{M}_{\mathbb{R}})$.

Theorem 4.7. *For $M = M^* = \mathbb{Q}$ with $\gamma_0 = 1$, we get*

$$\lim_{\alpha \rightarrow \infty} |\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^2 Q} W_{Q, \infty}(\Psi_\infty(s), g_l(\alpha)) = \tilde{\Upsilon}_\infty(\gamma_0) \Gamma_{1, m}(s - s_0) |\gamma|^s |2\gamma|^{\frac{1}{2}(s - s_0)}.$$

(Here: $\gamma = 2Q$).

For arbitrary n , with $m > 2n$ and $s = s_0$ (holomorphic special value), we get

$$|\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^1 Q \cdot \gamma_0^{-1}} W_{Q, \infty}(\Psi_\infty(s_0), g_l(\alpha)) = \begin{cases} \tilde{\Upsilon}_\infty(\gamma_0) \Gamma_{n, m}(0) |\gamma|^{s_0} & Q > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\gamma|$ is computed via the canonical measure μ_{γ_0} .

For the definition of $\Gamma_{n,m}(s)$ see 2.6.

Remark 4.8. *The limit in the first equation will be related to integrals of Borchers forms in 11.6.*

The second equation is expected because for a positive definite space $L_{\mathbb{R}}$, we have the Gaussian $\varphi_{\infty} \in S(L_{\mathbb{R}} \otimes M_{\mathbb{R}}^)$, defined by*

$$\varphi_{\infty}(\alpha) = \exp(-2\pi\alpha^! Q_L \cdot \gamma_0^{-1}).$$

It satisfies $\Phi(\varphi_{\infty}) = \Psi_{\infty, \frac{m}{2}}$, therefore we get (cf. equation (4))

$$\begin{aligned} W_{Q,\infty}(\Psi_{\infty}(s_0), 1) &= \tilde{\Upsilon}_{\infty}(\gamma_0) \int_{I(M^Q, L)(\mathbb{R})} \varphi_{\infty}(\alpha) \mu_{L, \gamma_0}(\alpha) \\ &= \tilde{\Upsilon}_{\infty}(\gamma_0) \exp(-2\pi Q \cdot \gamma_0^{-1}) \Gamma_{n,m}(0) |\gamma|^{\frac{m-n-1}{2}}, \end{aligned}$$

in accordance with the theorem. However, below we will give a different proof of the formula, based on Shimura's work [31].

Proof of theorem 4.7. The Iwasawa decomposition of the argument of $\Psi_{\infty, l}$ in the Whittaker integral can be expressed as

$$d(\gamma_0)u(\beta) = u(\Delta^2 \beta) g_l(\Delta) k(\gamma_0 \Delta \beta + i \gamma_0 \Delta \gamma_0^{-1}),$$

where $\Delta = (\sqrt{1 + (\beta \gamma_0)^2})^{-1}$. It satisfies ${}^t \Delta = \gamma_0 \Delta \gamma_0^{-1}$. Hence we get:

$$W_{Q,\infty}(\Psi_{\infty}(s), 1) = \int_{(M \otimes M)_{\mathbb{R}}^s} |\Delta|^{s + \frac{n+1}{2}} \chi_l(\gamma_0 \Delta \beta + i {}^t \Delta) \chi(Q \cdot \beta) \mu_{\gamma_0}(\beta)$$

and after choosing an orthonormal basis for γ_0 :

$$W_{Q,\infty}(\Psi_{\infty}(s), 1) = \int_{(\mathbb{R}^n \otimes \mathbb{R}^n)^s} \det(X + i)^{-a} \det(X - i)^{-b} e^{-2\pi i \operatorname{tr}(\frac{1}{2} X \gamma)} dX,$$

where γ is the bilinear form associated with Q (expressed in the chosen basis in the above formula).

with $a = \frac{1}{2}(s + \frac{n+1}{2} + l)$ and $b = \frac{1}{2}(s + \frac{n+1}{2} - l)$. Here $\det(X + i)^{-a} = e^{-a(\frac{\pi}{2} \pi i + \log(\det(1 - iX)))}$ and $\det(X - i)^{-b} = e^{-b(-\frac{\pi}{2} \pi i + \log(\det(1 + iX)))}$, where \log is the main branch of logarithm. dX is the measure defined in 2.4 for the standard basis, without the determinant factor. It is the same as used in [31].

Shimura [loc. cit.] denotes this function $(\xi(1, \frac{\gamma}{2}; a, b)$ in his notation) in analogy to the one dimensional case a **confluent hypergeometric function**. Furthermore [31, 1.29, 3.1K, 3.3] if Q is positive definite, the RHS equals

$$e^{-\pi \operatorname{tr} \gamma + i\pi \frac{n}{2}(b-a)} \pi^{na+nb} 2^n \det(\gamma)^{a+b-\frac{n+1}{2}} \Gamma_n(a)^{-1} \Gamma_n(b)^{-1} \zeta(2\pi\gamma; a, b),$$

where

$$\zeta(Z; a, b) = \int_{X > 0} e^{-\operatorname{tr} ZX} \det(X + 1)^{a-\frac{n+1}{2}} \det(X)^{b-\frac{n+1}{2}} dX.$$

In the 1 dimensional case, this gives

$$\zeta(Z; a, b) = \Gamma(b)U(b, a + b, Z),$$

where $U(k, l, Z)$ is a solution of the classical hypergeometric differential equation

$$Zf''(Z) + (l - Z)f'(Z) - kf(Z) = 0,$$

see [1, §13]. We have: $U(b, a + b, Z) = Z^{-b}(1 + O(|Z|^{-1}))$ [loc. cit.].

Therefore the ‘value at ∞ ’ for $l = \frac{m}{2}$ is computed as

$$\begin{aligned} & \lim_{\alpha \rightarrow \infty} |\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^2 Q} W_{Q, \infty}(\Psi_\infty(s), g_l(\alpha)) \\ &= \lim_{\alpha \rightarrow \infty} |\alpha|^{-s+1-\frac{m}{2}} e^{\pi\alpha^2 \gamma} W_{\alpha! Q, \infty}(\Psi_\infty(s), 1) \\ &= \lim_{\alpha \rightarrow \infty} |\alpha|^{-s+1-\frac{m}{2}} e^{-i\pi\frac{m}{4}} \pi^{s+1} 2|\alpha^2 \gamma|^s \Gamma_1(a)^{-1} \Gamma_1(b)^{-1} \zeta(2\pi\alpha^2 \gamma, a, b) \\ &= e^{-i\pi\frac{m}{4}} \pi^{s+1} 2|\gamma|^{\frac{1}{2}(s+1+\frac{m}{2})} \Gamma_1(\frac{1}{2}(s+1+\frac{m}{2}))^{-1} (2\pi)^{-\frac{1}{2}(s+1-\frac{m}{2})} \\ &= \tilde{\Upsilon}_\infty(\gamma_0) \Gamma_{1,m}(s-s_0) |\gamma|^{\frac{1}{2}(s+s_0)} 2^{-\frac{1}{2}(s-s_0)}, \end{aligned}$$

where $s_0 = \frac{m}{2} - 1$.

This proves the first formula of the theorem.

For the higher dimensional case, but for $s = \frac{m-n-1}{2}$ and $l = \frac{m}{2}$, we have $a = \frac{1}{2}m$ and $b = 0$ (holomorphic Eisenstein series) and we get for $m > 2n$:

$$\begin{aligned} & \det(\alpha)^{-\frac{m}{2}} e^{\pi \operatorname{tr}({}^t \alpha \gamma \alpha)} W_{Q, \infty}(\Psi_\infty(s), g_l(\alpha)) = \\ & \det(\alpha)^{-\frac{m}{2}} e^{\pi \operatorname{tr}({}^t \alpha \gamma \alpha)} \det(\alpha)^{\frac{m}{2}} \int_{(\mathbb{R}^n \otimes \mathbb{R}^n)^s} \det(X + \alpha^t \alpha i)^{-\frac{1}{2}m} e^{-\pi i \operatorname{tr}(X \gamma)} dX \\ &= e^{\pi \operatorname{tr}({}^t \alpha \gamma \alpha)} e^{-i\pi\frac{nm}{4}} (2\pi)^{\frac{1}{2}nm} \int_{(\mathbb{R}^n \otimes \mathbb{R}^n)^s} \det(2\pi i X + 2\pi \alpha^t \alpha)^{-\frac{1}{2}m} e^{\pi i \operatorname{tr}(X \gamma)} dX \\ &= \begin{cases} e^{-i\pi\frac{nm}{4}} \Gamma_{n,m}(0) \det(\gamma)^{\frac{m-n-1}{2}} & \gamma > 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

using [31, p. 174, (1.23)]. □

5. The local orbit equation — non-Archimedean case

5.1. Let R be a \mathbb{Q}_p or \mathbb{R} . Let M , N and L be R -vector spaces with non-degenerate quadratic forms Q_M, Q_N, Q_L respectively.

Consider the composition map:

$$\mathrm{I}(N, M) \times \mathrm{I}(M, L) \rightarrow \mathrm{I}(N, L).$$

Fixing an α in $\mathrm{I}(N, M)$, we may identify the fibre of the resulting map

$$\begin{aligned} \mathrm{I}(M, L) &\rightarrow \mathrm{I}(N, L) \\ \delta &\mapsto \delta \circ \alpha \end{aligned} \tag{6}$$

over $\beta \in I(N, L)$ with $I(\text{im}(\alpha)^\perp, \text{im}(\beta)^\perp)$.

All recursive properties that we will investigate in this section follow formally from the following theorem:

Theorem 5.2. *The resulting fibration is compatible with the canonical measures. If $\dim(M) = \dim(N)$ this means that the map (6) preserves volume.*

Proof. Let Q_M be the chosen form on M . By assumption, $\alpha^!Q_M$ is the chosen form Q_N .

We decompose M in $M_1 = \alpha(N)$ and $M_2 = M_1^\perp$ (orthogonal with respect to Q_M).

Decompose $\text{Sym}^2(M^*)$ with respect to Q_M :

$$\text{Sym}^2(M^*) = \text{Sym}^2(M_1^*) \oplus (M_1^* \otimes M_2^*) \oplus \text{Sym}^2(M_2^*),$$

where $\text{Sym}^2(M_1^*) = \text{Sym}^2(N^*)$ via α .

We get a commutative diagram of fibrations:

$$\begin{array}{ccccc} I(\alpha(N)^\perp, \beta(N)^\perp) & \xrightarrow{\quad \quad \quad} & M_2^* \otimes L & \xrightarrow{\delta \mapsto \begin{pmatrix} \delta^! Q_L \\ \langle \beta, \delta \rangle_L \end{pmatrix}} & \text{Sym}^2(M_2^*) \oplus (M_1^* \otimes M_2^*) \\ \downarrow & & \downarrow \text{incl.} + \beta \circ \text{pr}_1 & & \downarrow \text{incl.} + \beta^! Q_L \\ I(M^\gamma, L) & \xrightarrow{\quad \quad \quad} & M^* \otimes L & \xrightarrow{\delta \mapsto \delta^! Q_L} & \text{Sym}^2(M^*) \\ \downarrow \circ \alpha & & \downarrow \circ \alpha & & \downarrow \alpha^! \\ I(N^{\alpha^! \gamma}, L) & \xrightarrow{\quad \quad \quad} & N^* \otimes L & \xrightarrow{\delta \mapsto \delta^! Q_L} & \text{Sym}^2(N^*) \end{array}$$

where β varies in $N^* \otimes L$ such that $\beta^! Q_L$ varies in a neighborhood of Q_N and γ varies in $\text{Sym}^2(M^*)$ such that $\beta^! Q_L = \alpha^! \gamma$. $I(M^\gamma, L)$ is the fibre of the map $\delta \mapsto \delta^! Q_L$ in the middle row over γ and $I(N^{\alpha^! \gamma}, L)$ is the fibre of the map $\delta \mapsto \delta^! Q_L$ in the bottom row above $\alpha^! \gamma$. $I(\alpha(N)^\perp, \beta(N)^\perp)$ in the top-left corner is the fibre of the composition with α above β .

In the diagram, the vertical middle and rightmost fibrations come from (splitting) exact sequences of vector spaces.

The dotted map is defined by commutativity of the diagram. First observe that the (underlying) vertical exact sequences of vector spaces are exact with respect to canonical measures on the various spaces associated with Q_M, Q_L and $\alpha^! Q_M = Q_N$ and the restrictions of Q_M to M_1, M_2 respectively. The induced measure on $I(\alpha(M)^\perp, \beta(N)^\perp)$ hence is described by the topmost *horizontal* fibration as well.

Decompose $M_2^* \otimes L = M_2^* \otimes (\beta(N) \oplus \beta(N)^\perp)$. The map $\delta \mapsto \langle \beta, \delta \rangle_{Q_L}$ is an isomorphism $(M_2^* \otimes \beta(N)) \cong (M_1^* \otimes M_2^*)$ and 0 on the other factor. This isomorphism preserves the canonical volume, if $\beta^! Q_L = Q_N$.

Letting γ vary only in $\text{Sym}^2(M_2^*)$ fixing the other projections to 0 in $M_1^* \otimes M_2^*$ and to Q_N in $\text{Sym}(M_1^*)$ (i.e. having also $\beta^! Q_L = Q_N$), we get an equivalent

topmost horizontal fibration:

$$I(\alpha(N)^{\perp\gamma}, \beta(N)^{\perp}) \hookrightarrow M_2^* \otimes \beta(N)^{\perp} \xrightarrow{\delta \mapsto \delta^! Q_L} \text{Sym}^2(M_2^*)$$

and the dotted map is equal to the canonical inclusion into $\alpha(N)^{\perp*} \otimes \beta(N)^{\perp}$, noting $\alpha(N)^{\perp*} = M_2^*$. The induced measure on $I(\alpha(N)^{\perp\gamma}, \beta(N)^{\perp})$, however, is *by definition* the canonical measure. \square

Consider the situation of 5.1 for $R = \mathbb{Q}_p$. Consider $N_{\mathbb{Q}_p}$ via $\alpha \in N_{\mathbb{Q}_p}^* \otimes M_{\mathbb{Q}_p}$ as subspace of $M_{\mathbb{Q}_p}$. Choose lattices $N_{\mathbb{Z}_p} \subset M_{\mathbb{Z}_p}$ such that $M_{\mathbb{Z}_p} = N_{\mathbb{Z}_p} \oplus N'_{\mathbb{Z}_p}$. Choose a third lattice $L_{\mathbb{Z}_p}$ as well and assume that $Q_L \in \text{Sym}^2(L_{\mathbb{Z}_p}^*)$. Choose a coset $\kappa \in (L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}) \otimes M_{\mathbb{Z}_p}^*$. Denote $\kappa = \kappa_N \oplus \kappa_{N'}$ the corresponding decomposition of κ .

Corollary 5.3 (Kitaoka's formula).

$$\mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa) = \sum_{\substack{SO'(L_{\mathbb{Z}_p})\alpha \subseteq I(N, L)(\mathbb{A}^{(\infty)}) \cap \kappa_N \\ \kappa_{N'} \cap \alpha(N)_{\mathbb{Q}_p}^{\perp} \otimes (N'_{\mathbb{Q}_p})^* \neq \emptyset}} \mu_p(L_{\mathbb{Z}_p}, N_{\mathbb{Z}_p}, \kappa_N; \alpha) \mu_p(\alpha(N)_{\mathbb{Z}_p}^{\perp}, N'_{\mathbb{Z}_p}, \kappa_{N'}),$$

where $\kappa_{N'}$ is considered as an element of $((\alpha(N)_{\mathbb{Z}_p}^{\perp})^*/\alpha(N)_{\mathbb{Z}_p}^{\perp}) \otimes (N'_{\mathbb{Z}_p})^*$ via intersection with $\alpha(N)_{\mathbb{Q}_p}^{\perp} \otimes (N'_{\mathbb{Q}_p})^*$ and $\mu_p(L_{\mathbb{Z}_p}, N_{\mathbb{Z}_p}, \kappa_N; \alpha)$ is the volume of the SO' -orbit of α .

Proof. Integrate the characteristic function of κ over $I(M, L)(\mathbb{Q}_p)$ with respect to the canonical measure.

The intersections of κ with the fibres of the (restriction) map $I(M, L)(\mathbb{Q}_p) \rightarrow I(N, L)(\mathbb{Q}_p)$ over $\beta \in I(N, L)(\mathbb{Q}_p)$ can be identified with those isometries in $I(N', \alpha(N)^{\perp})(\mathbb{Q}_p)$ which lie in $\kappa_{N'}$.

The volume of these sets is constant, for conjugated α . Hence the formula follows from Theorem 5.2. \square

This is a slight generalization of the case $\kappa = L_{\mathbb{Z}_p} \otimes M_{\mathbb{Z}_p}^*$ where we get explicitly (for the representation densities β instead of the μ 's):

Corollary 5.4 ([12, Theorem 5.6.2]).

$$\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}) = \sum_i \left(\frac{d_p(K_i^{\perp})}{d_p(N_{\mathbb{Z}_p})d_p(L_{\mathbb{Z}_p})} \right)^{\frac{n-k}{2}} \beta(L_{\mathbb{Z}_p}, N_{\mathbb{Z}_p}; K_i) \beta(K_i^{\perp}, N'_{\mathbb{Z}_p}),$$

where we numbered the orbits and wrote K_i for $\alpha_i(N_{\mathbb{Z}_p})$ and k is the dimension of N . Here one is allowed to take SO -orbits, too.

Proof. The formula of the last corollary yields:

$$\begin{aligned}
& d_p(M_{\mathbb{Z}_p})^{\frac{n-m+1}{2}} d_p(L_{\mathbb{Z}_p})^{\frac{n}{2}} \beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}) \\
= & \sum_i d_p(N_{\mathbb{Z}_p})^{\frac{k-m+1}{2}} d_p(L_{\mathbb{Z}_p})^{\frac{k}{2}} \beta_p(L_{\mathbb{Z}_p}, N_{\mathbb{Z}_p}; K_i) \\
& \cdot d_p(N'_{\mathbb{Z}_p})^{\frac{n-m+1}{2}} d_p(K_i^\perp)^{\frac{n-k}{2}} \beta_p(K_i^\perp, N'_{\mathbb{Z}_p})
\end{aligned}$$

and after a reordering of the discriminant factors the statement follows. \square

5.5. Consider again an $\alpha \in \mathbf{I}(M, L)$ and the resulting map:

$$\begin{aligned}
\mathbf{I}(L, L) & \rightarrow \mathbf{I}(M, L), \\
\delta & \mapsto \delta \circ \alpha.
\end{aligned}$$

The fibre of this map over $\beta \in \mathbf{I}(M, L)$ can again be identified with $\mathbf{I}(\alpha(L)^\perp, \beta(L)^\perp)$. In particular the fibre over α is an orthogonal group again. We denote by $\mathrm{SO}(L_{\mathbb{Q}_p})$, respectively $\mathrm{SO}(\alpha(L_{\mathbb{Q}_p})^\perp)$ the corresponding *special* orthogonal groups. Choose lattices $L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}$, such that $Q_L \in \mathrm{Sym}^2(L_{\mathbb{Z}_p})$. Let κ be a class in $(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}) \otimes M_{\mathbb{Z}_p}^*$.

Consider the discriminant kernel $\mathrm{SO}'(L_{\mathbb{Z}_p}) \subseteq \mathrm{SO}(L_{\mathbb{Q}_p})$ of $L_{\mathbb{Z}_p}$, i.e. the kernel of the induced homomorphism $\mathrm{SO}(L_{\mathbb{Z}_p}) \rightarrow \mathrm{Aut}(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p})$.

It is a compact open subgroup. Consider the orbits of $\mathrm{SO}'(L_{\mathbb{Z}_p})$ acting on $\mathbf{I}(M, L) \cap \kappa$. The fibres over an orbit $\mathrm{SO}'(L_{\mathbb{Z}_p})\alpha$ all can be identified with $\mathrm{SO}'(\alpha_i(M)^\perp \cap L_{\mathbb{Z}_p})$. This follows from Lemma A.1 in the appendix.

Corollary 5.6 (elementary orbit equation).

$$\mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p}))^{-1} \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa) = \sum_{\mathrm{SO}'(L_{\mathbb{Z}_p})\alpha \subseteq \mathbf{I}(M, L)(\mathbb{Q}_p) \cap \kappa} \mathrm{vol}(\mathrm{SO}'(\alpha(M)_{\mathbb{Z}_p}^\perp))^{-1},$$

with the convention of 2.5.

Proof. From the theorem

$$\mathrm{vol}(\mathrm{SO}'(\alpha(M)_{\mathbb{Z}_p}^\perp) \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; \alpha) = \mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p})),$$

follows immediately, where in $\mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; \alpha)$ we mean the volume of the orbit of α .

Summed up over all orbits, we get the required statement. \square

5.7. Let $H_{\mathbb{Z}_p}$ be a hyperbolic plane, $\varphi \in S(L_{\mathbb{Q}_p} \otimes M_{\mathbb{Q}_p}^*)$ and form

$$\varphi^{(s)} := \varphi \otimes \chi_{H_{\mathbb{Z}_p}^s \otimes M_{\mathbb{Z}_p}^*} \in S((L_{\mathbb{Q}_p} \perp H_{\mathbb{Q}_p}^s) \otimes M_{\mathbb{Q}_p}^*).$$

We are interested in the function

$$s \mapsto \mu_p(L \perp H^s, M_{\mathbb{Z}_p}, \varphi^{(s)}),$$

for $s \in \mathbb{Z}_{\geq 0}$. We will denote the so constructed ‘interpolation’ of $\mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa)$ by $\mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s)$. This construction is motivated by the natural continuation of Fourier coefficients of Eisenstein series, see 4.5. Note that the μ_p as a function of s now depend on the lattice $M_{\mathbb{Z}_p}$, too, and not only on the choice of a $\varphi \in S(L_{\mathbb{Q}_p} \otimes M_{\mathbb{Q}_p}^*)$.

5.8. We will now construct an ‘interpolation’ of the volume of the orthogonal group, or, more precisely, of its discriminant kernel such that the above orbit equation remains true as an identity of functions in s .

Assume again, Q_M, Q_L non-degenerate. As a first step, we have

$$\begin{aligned} & \text{vol}(\text{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}^s))^{-1} \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s) \\ &= \sum_{\text{SO}'(L_{\mathbb{Z}_p}) \alpha \subseteq \text{I}(M, L)(\mathbb{Q}_p) \cap \kappa} \text{vol}(\text{SO}'(\alpha(M_{\mathbb{Z}_p})^\perp))^{-1}. \end{aligned}$$

The stability of these orbits for arbitrary s in this formula will be shown (at least for $p \neq 2$) in Lemma 5.13 below. This occurs at least, if L splits n hyperbolic planes. Hence the equation determines μ_p in any case for sufficiently large s , if the right hand side is interpreted appropriately.

Definition 5.9. We introduce the following notation:

$$\begin{aligned} \lambda_p(L_{\mathbb{Z}_p}; s) &:= \frac{\text{vol}(\text{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}^s))}{\prod_{i=1}^s (1 - p^{-2i})}, \\ \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s) &:= \text{vol}(\text{I}(M, L \oplus H_{\mathbb{Z}_p}^s)(\mathbb{Q}_p) \cap \kappa \oplus H_{\mathbb{Z}_p}^s \otimes M_{\mathbb{Z}_p}^*) \\ &= |d(L_{\mathbb{Z}_p})|^{\frac{n}{2}} |d(M_{\mathbb{Z}_p})|^{-s + \frac{1+n-m}{2}} \beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s), \end{aligned}$$

cf. also Lemma 3.7. Here all volumes are understood to be calculated w.r.t. the canonical measures (2.4).

We have the following **orbit equation**:

Theorem 5.10. Let $p \neq 2$. Assume, that $L_{\mathbb{Z}_p}$ splits n hyperbolic planes. We have

$$\lambda_p(L_{\mathbb{Z}_p}; s)^{-1} \cdot \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s) = \sum_{\text{SO}'(L_{\mathbb{Z}_p}) \alpha \subseteq \text{I}(M, L)(\mathbb{Q}_p) \cap \kappa} \lambda_p(\alpha(M_{\mathbb{Z}_p})^\perp; s)^{-1}.$$

Remark 5.11. We will show in 8.1 that, if $s \geq 1$, $\lambda_p(L_{\mathbb{Z}_p}; s)$ is an actually quite simple polynomial in p^{-s} . It is easy to see from the explicit formula that for any sublattice $L'_{\mathbb{Z}_p} \subseteq L_{\mathbb{Z}_p}$, $\lambda(L'_{\mathbb{Z}_p}; s)$ divides $\lambda(L_{\mathbb{Z}_p}; s)$ as a polynomial in p^{-s} . Hence the $\mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s)$ are polynomials for sufficiently large s (s.t. orbits are stable), and hence the orbit equation makes sense for all $s \in \mathbb{C}$. The fact that the μ_p ’s are polynomials in p^{-s} for large s can also be proven using their equality with Whittaker functions (Theorem 4.5).

Observe: $\prod_{i=1}^s (1 - p^{-2i}) = (1 + p^{-s}) \text{vol}(\text{SO}(H^s))$ for $s \in \mathbb{N}$ (cf. 8.1).

In 8.1 the occurring functions will be calculated explicitly in a lot of cases. We now turn to the problem of stability of orbits:

Lemma 5.12. *Consider $L_{\mathbb{Z}_p} \oplus H^2 = L_{\mathbb{Z}_p} \oplus \mathbb{Z}_p^4$ (i.e. a space with quadratic form $Q(x_L, x_0, \dots, x_3) = Q_L(x_L) + x_0x_1 + x_2x_3$). Let $w = (w_L, w_0, \dots, w_3) \in L_{\mathbb{Z}_p} \oplus H^2$ be a vector with $Q(w) \neq 0$. It follows*

$$\langle w \rangle^\perp \equiv H \perp \Lambda.$$

Proof. We may assume w.l.o.g. that $\nu_p(w_0)$ is minimal among the $\nu_p(w_i)$. $\langle w \rangle^\perp$ is described by the equation $\langle w_L, x_L \rangle + w_0x_1 + w_1x_0 + w_2x_3 + w_3x_2 = 0$. The map $(x_2, x_3) \mapsto (0_L, 0, -\frac{w_2x_3 + w_3x_2}{w_0}, x_2, x_3)$ therefore is an isometric embedding of an hyperbolic plane into $\langle w \rangle^\perp$. The assertion now follows from Lemma A.2. \square

Let $\kappa \in (L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}) \otimes M_{\mathbb{Z}_p}^*$ and $\{\alpha_i\}$ be a set of representatives of the orbits under $\text{SO}'(L_{\mathbb{Z}_p})$ acting on $\text{I}(M, L)(\mathbb{Q}_p) \cap \kappa$.

Lemma 5.13 (stability of orbits). *Assume $M_{\mathbb{Z}_p}$ has dimension n , $p \neq 2$ and $L_{\mathbb{Z}_p}$ splits n hyperbolic planes.*

Then $\{\alpha_i\}$ is a set of representatives of the $\text{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}^s)$ -orbits in $\text{I}(M, L \perp H^s)(\mathbb{Q}_p) \cap (\kappa \oplus (H_{\mathbb{Z}_p}^s \otimes M_{\mathbb{Z}_p}^))$ for all s .*

Proof. We begin by showing that, if $\alpha_i = g\alpha_j$ for some $g \in \text{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}^s)$, then we have $\alpha_i = g'\alpha_j$ for some $g' \in \text{SO}'(L_{\mathbb{Z}_p})$ as well. We have

$$\alpha_i(M_{\mathbb{Z}_p})^\perp = H_{\mathbb{Z}_p}^s \perp \alpha_i(M_{\mathbb{Z}_p})^{\perp L_{\mathbb{Z}_p}}.$$

Since the form is integral in $L_{\mathbb{Z}_p}$, we have according to Lemma A.3:

$$\alpha_i(M_{\mathbb{Z}_p})^\perp = g(H_{\mathbb{Z}_p}^s) \perp \Lambda_{\mathbb{Z}_p}$$

(because $H_{\mathbb{Z}_p}^s \perp \alpha_j(M_{\mathbb{Z}_p})$.) Hence (Lemma A.3 — here $p \neq 2$ is used), there is an isometry $\alpha_i(M_{\mathbb{Z}_p})^\perp$, which maps $g(H_{\mathbb{Z}_p}^s)$ to $H_{\mathbb{Z}_p}^s$ and lies in $\text{SO}'(\alpha_i(M_{\mathbb{Z}_p})^\perp)$. Hence it lifts to an isometry in $\text{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}^s)$, which fixes $M_{\mathbb{Z}_p}$ pointwise (Lemma A.1). Composition with g yields the required g' .

Secondly, let an isometry $\alpha : M_{\mathbb{Z}_p} \rightarrow L_{\mathbb{Z}_p} \perp H_{\mathbb{Z}_p}^s$ be given. We have to show that it is mapped by an element in $\text{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}^s)$ to any of the α_i . It clearly suffices (induction on s) to consider the case $s = 1$.

We proceed by induction on n and first prove the case $n = 1$: By Lemma 5.12 $\alpha_i(M_{\mathbb{Z}_p})^\perp$ (\perp with respect to $L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}$) splits an hyperbolic plane because by assumption $L_{\mathbb{Z}_p}$ splits already one. Then apply Lemma A.3. We get

$$\alpha(M_{\mathbb{Z}_p})^\perp = \Lambda_{\mathbb{Z}_p} \perp \Lambda'_{\mathbb{Z}_p},$$

with $\Lambda_{\mathbb{Z}_p} \cong H_{\mathbb{Z}_p}$.

In addition, we have (again Lemma A.3)

$$L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p} = \Lambda_{\mathbb{Z}_p} \perp \Lambda_{\mathbb{Z}_p}^\perp,$$

hence (Lemma A.6) $\Lambda_{\mathbb{Z}_p}^\perp \cong L_{\mathbb{Z}_p}$ and the image of α of course lies in $\Lambda_{\mathbb{Q}_p}^\perp$ because $\Lambda_{\mathbb{Q}_p} \perp \alpha(M_{\mathbb{Q}_p})$. Now there is an isometry g , which maps $\Lambda_{\mathbb{Z}_p}^\perp$ to $L_{\mathbb{Z}_p}$ and $\Lambda_{\mathbb{Z}_p}$ to $H_{\mathbb{Z}_p}$ (even in $\mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p})$ — Lemma A.6). The image of $g \circ \alpha$ then lies in $L_{\mathbb{Q}_p}$ and the element of $\mathrm{SO}'(L_{\mathbb{Z}_p})$, which maps $g \circ \alpha$ to any α_i , lifts to an isometry $g' \in \mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p})$. Hence α is conjugated to α_i under $\mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p})$.

We now assume, that the statement has been proven for M up to dimension $n - 1$. We choose some splitting $M = N \oplus N^\perp$ with $\dim(N) = n - 1$ and accordingly decomposition $\kappa = \kappa_N + \kappa_{N^\perp}$. Let $\alpha : M_{\mathbb{Z}_p} \rightarrow L_{\mathbb{Z}_p} \perp H_{\mathbb{Z}_p}^s$ be given. Induction hypothesis shows, that w.l.o.g. $\alpha(N) \subset L_{\mathbb{Q}_p}$. Since $L_{\mathbb{Z}_p}$ splits an unrequired hyperbolic plane, we may even assume, that $\alpha(N)^\perp \cap L_{\mathbb{Z}_p}$ splits a hyperbolic plane, too. Hence we may apply the $n = 1$ case to $\alpha(N)^\perp \oplus H_{\mathbb{Z}_p}^s$ and M^\perp and observe that the constructed isometries in this step all lift by Lemma A.1. \square

We also get immediately an interpolated version of Kitaoka's formula:

Theorem 5.14. *Let $p \neq 2$. Assume, that $L_{\mathbb{Z}_p}$ splits n hyperbolic planes. With the notation of Corollary 5.3, we have*

$$\begin{aligned} \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s) = \\ \sum_{\mathrm{SO}'(L_{\mathbb{Z}_p}) \alpha \subseteq \mathrm{I}(M, L)(\mathbb{Q}_p) \cap \kappa} \mu_p(L_{\mathbb{Z}_p}, N_{\mathbb{Z}_p}, \kappa_N, \alpha; s) \mu_p(\alpha(N)_{\mathbb{Z}_p}^\perp, N'_{\mathbb{Z}_p}, \kappa_{N'}; s). \end{aligned}$$

The quantities $\mu_p(L_{\mathbb{Z}_p}, N_{\mathbb{Z}_p}, \kappa_N, \alpha; s)$ are by Theorem 5.2 equal to

$$\frac{\lambda_p(L_{\mathbb{Z}_p}; s)}{\lambda_p(\alpha(N)_{\mathbb{Z}_p}^\perp; s)},$$

hence they are polynomials in p^{-s} (cf. 5.11) and furthermore, the equation can be rewritten in the more symmetric form:

$$\begin{aligned} \lambda_p(L_{\mathbb{Z}_p}; s)^{-1} \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s) = \\ \sum_{\mathrm{SO}'(L_{\mathbb{Z}_p}) \alpha \subseteq \mathrm{I}(M, L)(\mathbb{Q}_p) \cap \kappa} \lambda_p(\alpha(N)_{\mathbb{Z}_p}^\perp; s)^{-1} \mu_p(\alpha(N)_{\mathbb{Z}_p}^\perp, N'_{\mathbb{Z}_p}, \kappa_{N'}; s). \end{aligned}$$

6. The local orbit equation — Archimedian case

Definition 6.1. *We will define factors at ∞ analogously to λ_p and μ_p (5.9).*

$$\begin{aligned} \lambda_\infty(L; s) &:= \Gamma_{m-1, m}(s), \\ \mu_\infty(L, M; s) &:= \Gamma_{n, m}(s), \end{aligned}$$

where, as usual, $n = \dim(M)$ and $m = \dim(L)$.

With this notation, we have also a (rather trivial) Archimedian analogue of the orbit equation (with only one orbit):

Theorem 6.2. *If Q_M is positive definite, we have*

$$\mu_\infty(L_\mathbb{Z}, M_\mathbb{Z}, \kappa; s) \cdot \lambda_\infty(L_\mathbb{Z}; s)^{-1} = \lambda_\infty(\alpha(L)^\perp; s)^{-1}.$$

Here α is any real embedding $M_\mathbb{R} \rightarrow L_\mathbb{R}$, with $\alpha^! Q_L = Q_M$. The above depends only on the respective dimensions and is formulated in dependence of L and M only in order to have the same shape than the non-Archimedian orbit equation (Theorem 5.10).

Furthermore in the positive definite case, we have, analogously to the non-Archimedian case:

$$\text{vol}(\text{I}(M, L)_\mathbb{R}) = \Gamma_{n,m}(0) \quad n \geq m$$

and in particular

$$\text{vol}(\text{SO}(M_\mathbb{R})) = \Gamma_{m-1,m}(0).$$

(for the canonical volumes 2.4), see Lemma 3.7.

7. Explicit formulæ, $n = 1$ case

In the case $n = 1$, the representation densities have been computed explicitly by Yang [35]:

7.1. Assume $p \neq 2$. Consider

$$L_{\mathbb{Z}_p} = \langle \varepsilon_1 p^{l_1}, \dots, \varepsilon_m p^{l_m} \rangle,$$

where $\varepsilon_i \in \mathbb{Z}_p^*$ and $l_i \in \mathbb{Z}_{\geq 0}$, $l_1 \leq \dots \leq l_m$. Assume, that $p^{-1}Q_L$ is not integral, i.e. we have $l_1 = 0$.

Denote:

$$\begin{aligned} L(k, 1) &:= \{1 \leq i \leq m \mid l_i - k < 0 \text{ is odd}\} \\ l(k, 1) &:= \#L(k, 1) \\ d(k) &:= k + \frac{1}{2} \sum_{l_i < k} (l_i - k) \\ v(k) &:= \left(\frac{-1}{p}\right)^{\lfloor \frac{l(k, 1)}{2} \rfloor} \prod_{i \in L(k, 1)} \left(\frac{\varepsilon_i}{p}\right). \end{aligned}$$

Theorem 7.2 ([35, Theorem 3.1]). *With this notation, we have*

$$\beta_p(L_{\mathbb{Z}_p}, \langle \alpha p^a \rangle, L_{\mathbb{Z}_p}; s) = 1 + R(\alpha p^a; p^{-s}),$$

where $\alpha \in \mathbb{Z}_p^*$, $a \in \mathbb{Z}_{\geq 0}$ and

$$\begin{aligned} R(\alpha p^a; X) &= (1 - p^{-1}) \sum_{\substack{0 < k \leq a \\ l(k,1) \text{ is even}}} v(k) p^{d(k)} X^k \\ &+ v(a+1) p^{d(a+1)} X^{a+1} \begin{cases} -p^{-1} & \text{if } l(a+1,1) \text{ is even,} \\ \left(\frac{\alpha}{p}\right) p^{-\frac{1}{2}} & \text{if } l(a+1,1) \text{ is odd.} \end{cases} \end{aligned}$$

We have:

$$\beta_p(L_{\mathbb{Z}_p}, < 0 >, L_{\mathbb{Z}_p}; s) = 1 + R(0; p^{-s}),$$

where

$$R(0; X) = (1 - p^{-1}) \sum_{\substack{k > 0 \\ l(k,1) \text{ is even}}} v(k) p^{d(k)} X^k.$$

Furthermore, still for $n = 1$, the representation densities have the following interpretations:

Lemma 7.3. *Let $s \in \mathbb{Z}_{\geq 0}$ and let κ be a coset in $L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}$.*

We have the following relation to representation numbers:

$$\beta_p(L_{\mathbb{Z}_p}, < q >, \kappa; s) = \#\Omega_{\kappa,q}(w) p^{w(1-m-s)} + (1 - p^{-s}) \sum_{j=0}^{w-1} \#\Omega_{\kappa,q}(j) p^{j(1-m-s)} \quad (7)$$

Here $\Omega_{\kappa,q}(j) = \{v \in L_{\mathbb{Z}/p^j\mathbb{Z}} \mid Q_M(v + \kappa) \equiv q \pmod{p^j}\}$ and w is a sufficiently large integer. (Explicitly: $w \geq 1 + 2\nu_p(2q \operatorname{ord}(\kappa))$ — the formula then does not depend on w .)

This can be written as follows ($\operatorname{Re} s > 1$):

$$\sum_l \frac{\#\Omega_{\kappa,q}(l)}{p^{l(m-1+s)}} = \frac{\beta_p(L_{\mathbb{Z}_p}, < q >, \kappa; s)}{1 - p^{-s}}. \quad (8)$$

If $m \geq 2$,

$$\int_{\kappa} |Q_L(v) - Q|^s dv = p^s + \beta(L_{\mathbb{Z}_p}, < Q >, \kappa; s+1) \frac{1 - p^s}{1 - p^{-s-1}}, \quad (9)$$

where dv is the translation invariant measure with $\operatorname{vol}(L_{\mathbb{Z}_p}) = 1$.

Proof. Formula (7) is obviously true for $s = 0$. Under the substitution $L \rightsquigarrow L \oplus H$ the left hand side becomes $\beta_p(L_{\mathbb{Z}_p}, < q >, \kappa; s+1)$ and the right hand side becomes the same expression for $s+1$, if we use the relation:

$$\#\Omega_{\kappa,q}(L \oplus H, r) = \sum_{\nu=0}^{r-1} p^{(r-\nu)m} (p^r - p^{r-1}) \#\Omega_{\kappa,q}(L, \nu) + \#\Omega_{\kappa,q}(L, r) p^r. \quad (10)$$

Proof of the relation: An explicit calculation shows:

$$\#\Omega_n(H, l) = \begin{cases} (\nu_p(n) + 1)(p^l - p^{l-1}) & \nu_p(n) < l, \\ l(p^l - p^{l-1}) + p^l & \nu_p(n) \geq l. \end{cases} \quad (11)$$

Hence:

$$\begin{aligned} & \#\Omega_{\kappa, q}(L + H, l) \\ &= \sum_{n \in \mathbb{Z}/p^l \mathbb{Z}} \#\Omega_{\kappa, q-n}(L, l) \#\Omega_n(H, l) \\ &= \sum_{\nu=0}^l \#\Omega_{p^\nu}(H, l) \sum_{\substack{n \in \mathbb{Z}/p^l \mathbb{Z} \\ \nu_p(n) = \nu}} \#\Omega_{\kappa, q-n}(L, l) \\ &= \sum_{\nu=0}^l \#\Omega_{p^\nu}(H, l) \left(\sum_{\substack{n \in \mathbb{Z}/p^l \mathbb{Z} \\ \nu_p(n) \geq \nu}} \#\Omega_{\kappa, q-n}(L, l) - \sum_{\substack{n \in \mathbb{Z}/p^l \mathbb{Z} \\ \nu_p(n) \geq \nu+1}} \#\Omega_{\kappa, q-n}(L, l) \right) \\ &= \sum_{\nu=0}^l \#\Omega_{p^\nu}(H, l) \left(p^{(l-\nu)m} \#\Omega_{\kappa, q}(L, \nu) - p^{(l-\nu-1)m} \#\Omega_{\kappa, q}(L, \nu+1) \right). \end{aligned}$$

From this the relation (10) follows. (8) is obtained by letting $w \rightarrow \infty$ since (7) does not depend on w .

For formula (9) observe that

$$\begin{aligned} \int_{\kappa} |Q_L(v) - q|^s \, dv &= \sum_{i=0}^{\infty} \left(\text{vol } \kappa \cap \{|Q_L - Q| \leq \frac{1}{p^i}\} - \text{vol } \kappa \cap \{|Q_L - Q| \leq \frac{1}{p^{i+1}}\} \right) \frac{1}{p^{is}} \\ &= 1 + \sum_{i=1}^{\infty} \text{vol } \kappa \cap \{|Q_L - Q| \leq \frac{1}{p^i}\} (1 - p^s) \frac{1}{p^{is}} \\ &= 1 + \sum_{i=1}^{\infty} \frac{\#\Omega_{\kappa, q}(i)}{p^{i(s+n)}} (1 - p^s) \\ &= 1 - (1 - p^s) + \sum_{i=0}^{\infty} \frac{\#\Omega_{\kappa, q}(i)}{p^{i(s+n)}} (1 - p^s). \end{aligned}$$

From this (9) follows using identity (8). \square

7.4. We will investigate the zeta function representation given in the theorem a bit further. It is convenient to write

$$\beta(L_{\mathbb{Z}_p}, < 0 >; s) = 1 + (1 - p^{-1})\delta(p^{-s}),$$

where δ is a rational function, which has, according to Theorem 7.2, the expansion:

$$\delta(X) = \sum_{\substack{k > 1 \\ l(k, 1) \equiv 0 \pmod{2}}} \nu(k) p^{d(k)} X^k \quad (2)$$

(with the local notation from Theorem 7.2). Here $\delta(0) = 0$.
Therefore:

$$\begin{aligned} E := \text{vol}\{x \in L_{\mathbb{Z}_p} | Q_L(v) \in \mathbb{Z}_p^*\} &= \lim_{s \rightarrow \infty} \int_{L_{\mathbb{Z}_p}} |Q_L(v)|^s \, dv \\ &= \lim_{X \rightarrow 0} \left(1 - \frac{1 + (1 - p^{-1})\delta(p^{-1}X)}{1 - p^{-1}X}\right) \\ &= (1 - p^{-1})(1 - \delta'(0)p^{-1}). \end{aligned}$$

with

$$\delta'(0) = \begin{cases} \nu(1)p^{d(1)} & l(1, 1) \equiv 0 \pmod{2}, \\ 0 & l(1, 1) \equiv 1 \pmod{2}. \end{cases}$$

Definition 7.5. We define the **normalized local zeta function** associated with L by

$$\zeta_p(L_{\mathbb{Z}_p}; s) := \frac{1}{E} \int_{L_{\mathbb{Z}_p}} |Q_L(v)|^{s-1} \, dv.$$

For two dimensional lattices, this coincides for example with the usual zeta function of the associated order in the associated quadratic field ($\frac{dv}{E|Q_L|}$ is the multiplicatively invariant measure for which $(\mathcal{O} \otimes \mathbb{Z}_p)^*$ has volume 1).

7.6. Here, we explicitly compute the zeta function for an arbitrary two dimensional lattice. This will be used in section 12.

Let $L_{\mathbb{Z}_p}$, $p \neq 2$ be a lattice with $Q_L \in \text{Sym}^2(I_{\mathbb{Z}_p}^*)$, such that $p^{-1}Q_L$ is not integral. The zeta function of such a 2 dimensional lattice depends only on the discriminant because it is invariant multiplication of the form by a scalar $\in \mathbb{Z}_p^*$. Hence we may assume w.l.o.g. $L_{\mathbb{Z}_p} = \mathbb{Z}_p^2$ with $Q_L : x \mapsto x_1^2 + \varepsilon p^l(x_2)^2$. With the notation of Yang (Theorem 7.2), we have:

$$\begin{aligned} L(k; 1) &= \begin{array}{lll} l \text{ even} & & l \text{ odd} \\ \{\} & 0 < k \leq l \text{ even} & \{\} \\ \{1\} & 0 < k \leq l \text{ odd} & \{1\} \\ \{\} & l < k \text{ even} & \{2\} \\ \{1, 2\} & l < k \text{ odd} & \{1\} \end{array} \\ d(k) &= \begin{cases} \frac{1}{2}k & k \leq l, \\ \frac{1}{2}l & l < k, \end{cases} \\ \nu(k) &= \begin{cases} (\frac{-\varepsilon}{p})^k & l < k, l \text{ even}, \\ 1 & k \leq l \text{ or } l \text{ odd}. \end{cases} \end{aligned}$$

Assume first $l = 0$, then (as expected):

$$\begin{aligned} \delta(X) &= \sum_{k \geq 1} \left(\frac{-\varepsilon}{p}\right)^k X^k = \frac{(\frac{-\varepsilon}{p})X}{1 - (\frac{-\varepsilon}{p})X}, \\ \zeta_p(L_{\mathbb{Z}_p}; s) &= \frac{(1 - p^{-1})(1 + (1 - (pX)^{-1})\delta(X))}{(1 - X)E} = \frac{1}{(1 - X)(1 - (\frac{-\varepsilon}{p})X)}. \end{aligned}$$

For l odd, the above yields:

$$\begin{aligned}\delta(X) &= \sum_{k'=1}^{\frac{l-1}{2}} p^{k'} X^{2k'} = (pX)^2 \frac{1 - (pX^2)^{\frac{l-1}{2}}}{1 - (pX)^2}, \\ \zeta_p(L_{\mathbb{Z}_p}; s) &= \frac{1 - (pX^2)^{\frac{l+1}{2}} - X + X(pX^2)^{\frac{l-1}{2}}}{(1-X)(1-pX^2)}.\end{aligned}$$

For $l \geq 2$, even, it yields:

$$\begin{aligned}\delta(X) &= \sum_{k=l}^{\infty} \left(\frac{-\varepsilon}{p}\right)^k p^{\frac{1}{2}l} X^k + \sum_{k'=1}^{\frac{l}{2}-1} p^{k'} X^{2k'} \\ &= p^{\frac{1}{2}l} X^l \frac{1}{1 - (\frac{-\varepsilon}{p})X} + pX^2 \frac{1 - (pX^2)^{\frac{l}{2}-1}}{1 - pX^2}, \\ \zeta_p(L_{\mathbb{Z}_p}, s) &= \frac{p^{\frac{1}{2}l} X^l - p^{\frac{l}{2}-1} X^{l-1}}{(1-X)(1 - (\frac{-\varepsilon}{p})X)} + \frac{1 - (pX^2)^{\frac{l}{2}} - X + X(pX^2)^{\frac{l}{2}-1}}{(1-X)(1-pX^2)}.\end{aligned}$$

8. Explicit formulæ, general case

In this section, we will compute the functions μ and λ (cf. Definiton 5.9) explicitly in special cases. The expression for λ is quite general and can in principle be used to compute it for all lattices.

Theorem 8.1. *Assume $p \neq 2$.*

1. *Let $s \in \mathbb{N}$. $\text{vol}(\text{SO}(H^s)) = (1 - p^{-s}) \cdot \prod_{i=1}^{s-1} (1 - p^{-2i})$.*
2. *Let $s \in \mathbb{Z}_{\geq 0}$. $\lambda(L_{\mathbb{Z}_p} \perp H; s) = (1 - p^{-2s-2}) \cdot \lambda(L_{\mathbb{Z}_p}; s+1)$.*
3. *Let $s \in \mathbb{Z}_{\geq 0}$. Let $L_{\mathbb{Z}_p} = \langle \varepsilon_1, \dots, \varepsilon_k \rangle \perp L'_{\mathbb{Z}_p}$, where $\varepsilon_i \in \mathbb{Z}_p^*$ and $p^{-1}Q_L$ is integral on $L'_{\mathbb{Z}_p}$. Assume $k > 1$. Let $\varepsilon := (-1)^{\frac{k}{2}} \prod_{i=1}^k \varepsilon_i$, if k is even. Then we have*

$$\frac{\lambda(L_{\mathbb{Z}_p}; s)}{\lambda(L_{\mathbb{Z}_p}; 0)} = |D|_p^s \prod_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} \frac{1 - p^{-2i-2s}}{1 - p^{-2i}} \begin{cases} 1 & k \equiv 1 \pmod{2}, \\ \frac{1 - (\frac{\varepsilon}{p})p^{-\frac{k}{2}-s}}{1 - (\frac{\varepsilon}{p})p^{-\frac{k}{2}}} & k \equiv 0 \pmod{2}. \end{cases}$$

In particular $\lambda(L_{\mathbb{Z}_p}; s)$ is a (quite simple) polynomial in p^{-s} .

4. *Let $s \in \mathbb{Z}_{\geq 0}$. For a unimodular lattice of discriminant $2^m \varepsilon$ and $\varepsilon' \in \mathbb{Z}_p^*$ we have:*

$$\mu(L_{\mathbb{Z}_p}, \langle \varepsilon' \rangle; s) = \begin{cases} (1 - (\frac{(-1)^{\frac{m}{2}} \varepsilon}{p})p^{-\frac{m}{2}-s}) & m \equiv 0 \pmod{2}, \\ (1 + (\frac{(-1)^{\frac{m-1}{2}} \varepsilon \varepsilon'}{p})p^{-\frac{m-1}{2}-s}) & m \equiv 1 \pmod{2}. \end{cases}$$

5. Let $s \in \mathbb{Z}_{\geq 0}$. For a unimodular lattice of discriminant $2^m \varepsilon$ we have:

$$\lambda(L_{\mathbb{Z}_p}; s) = \prod_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} (1 - p^{-2i-2s}) \begin{cases} (1 - (\frac{(-1)^{\frac{m}{2}} \varepsilon}{p}) p^{-\frac{m}{2}-s}) & m \equiv 0 \pmod{2}, \\ 1 & m \equiv 1 \pmod{2}. \end{cases}$$

6. Let $s \in \mathbb{Z}_{\geq 0}$. For a lattice with $L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}$ cyclic of order $p^\nu \neq 1$ and dimension $m \geq 2$, we may assume $L = \mathbb{Z}_p^m$, $Q_L(x) = \sum_{j=1}^{m-1} \varepsilon_j x_j^2 + p^\nu \varepsilon_m x_m^2$. Denote $\varepsilon = (-1)^{\frac{m-1}{2}} \prod_{j=1}^{m-1} \varepsilon_j$ if m is odd. We have

$$\lambda(L_{\mathbb{Z}_p}; s) = |p^\nu|_p^{s + \frac{m-1}{2}} \prod_{i=1}^{\lfloor \frac{m}{2} \rfloor - 1} (1 - p^{-2i-2s}) \begin{cases} 1 & m \equiv 0 \pmod{2}, \\ 1 - (\frac{\varepsilon}{p}) p^{-\frac{m-1}{2}-s} & m \equiv 1 \pmod{2}. \end{cases}$$

Proof. 1. According to Kitaoka's formula (cf. Theorem 5.3), we have

$$\text{vol}(\text{SO}(H^s)) = \beta_p(H^s, 1) \beta_p(H^{s-1} \perp \langle -1 \rangle, -1) \text{vol}(\text{SO}(H^{s-1})).$$

Theorem 7.2 yields $\beta_p(H^s, 1) = 1 - p^{-s}$ and $\beta_p(H^{s-1} \perp \langle -1 \rangle, 1) = 1 + p^{-s+1}$. Furthermore, we have $\text{vol}(\text{SO}(H)) = \beta_p(H, 1) = 1 - p^{-1}$.

2. follows immediately from the definition of λ and 1.

3. Let $L_{\mathbb{Z}_p}$ be a lattice, and $S = \langle \alpha_1, \alpha_2 \rangle$ a unimodular plane (e.g. a hyperbolic one). Using the (elementary) orbit equation, Theorem 5.6 and Theorem 5.3, we get

$$\begin{aligned} \text{vol}(\text{SO}'(L_{\mathbb{Z}_p} \perp S)) &= |D|^{\frac{1}{2}} \beta_p(L_{\mathbb{Z}_p} \perp S, \alpha_2) |D|^{\frac{1}{2}} \beta_p(L_{\mathbb{Z}_p} \perp \langle \alpha_1 \rangle, \alpha_1) \\ &\quad \cdot \text{vol}(\text{SO}'(L_{\mathbb{Z}_p})). \end{aligned} \tag{12}$$

Hence, we have to apply Theorem 7.2 to forms of the shape

$$L'_{\mathbb{Z}_p} = \langle \varepsilon_1, \dots, \varepsilon_{k'}, p^{\nu_{k'+1}} \varepsilon_{k'+1}, \dots, p^{\nu_m} \varepsilon_m \rangle.$$

We get

$$\beta_p(L'_{\mathbb{Z}_p}, \varepsilon'; s) = 1 + v(1) p^{d(1)} p^{-s} \begin{cases} -p^{-1} & l(1, 1) \equiv 0 \pmod{2}, \\ (\frac{-\varepsilon'}{p}) p^{-\frac{1}{2}} & l(1, 1) \equiv 1 \pmod{2}. \end{cases}$$

We have

$$\begin{aligned} l(1, 1) &= k' \\ d(1) &= 1 - \frac{1}{2} k' \\ v(1) &= \left(\frac{-1}{p}\right)^{\lfloor \frac{k'}{2} \rfloor} \prod_{i=1}^{k'} \left(\frac{\varepsilon_i}{p}\right) \end{aligned}$$

Hence

$$\beta_p(L'_{\mathbb{Z}_p}, \varepsilon'; s) = \begin{cases} 1 - \left(\frac{(-1)^{\frac{k'}{2}} \prod_{i=1}^{k'} \varepsilon_i}{p}\right) p^{-\frac{k'}{2}-s} & k' \equiv 0 \pmod{2}, \\ 1 + \left(\frac{(-1)^{\frac{k'-1}{2}} \prod_{i=1}^{k'} \varepsilon_i \varepsilon'}{p}\right) p^{-\frac{k'-1}{2}-s} & k' \equiv 1 \pmod{2}. \end{cases}$$

Applying this to $L'_{\mathbb{Z}_p} = L_{\mathbb{Z}_p} \perp S$ and $L'_{\mathbb{Z}_p} \perp \langle \alpha \rangle$, we get the result for k odd. For k even write $L_{\mathbb{Z}_p} = L'_{\mathbb{Z}_p} \perp \langle \alpha \rangle$, use

$$\text{vol}(\text{SO}'(L'_{\mathbb{Z}_p})) = |D|^{\frac{1}{2}} \beta_p(L'_{\mathbb{Z}_p}, \alpha) \cdot \text{vol}(\text{SO}'(L_{\mathbb{Z}_p})). \quad (13)$$

twice and the k odd part. Recall (Lemma A.6) that vectors of length $\alpha \in \mathbb{Z}_p^*$ form one orbit under SO' , as long as the lattice in question splits a unimodular plane, otherwise there are 2 orbits of equal volume.

4. This is Siegel's formula, a special case of Theorem 7.2.

5. Follows from 4. and the orbit equation, Theorem 5.10.

6. Follows from 3. and the following calculation for $m \geq 2$ (which follows easily from equations (12) and (13) and the fact $\text{vol}(\text{SO}'(\langle x \rangle)) = 1$). Observe, that there are 2 orbits (of equal volume) of vectors of length β in a lattice $\langle \alpha p^\nu, \beta \rangle$.

$$\lambda(L_{\mathbb{Z}_p}; 0) = |p^\nu|^{\frac{m-1}{2}} \prod_{i=1}^{\lfloor \frac{m}{2} \rfloor - 1} (1 - p^{-2i}) \begin{cases} 1 & m \equiv 0 \pmod{2}, \\ 1 - \left(\frac{\varepsilon}{p}\right) p^{-\frac{m-1}{2}} & m \equiv 1 \pmod{2}. \end{cases}$$

Here $\varepsilon = (-1)^{\frac{m-1}{2}} \prod_{i=1}^{m-1} \varepsilon_i$. □

Without proof, we give here some calculations in the case $p = 2$. We plan to extend all results of this paper to the case $p = 2$ soon.

Theorem 8.2.

1. Let $s \in \mathbb{N}$. $\text{vol}(\text{SO}(H^s)) = (1 - 2^{-s}) \cdot \prod_{i=1}^{s-1} (1 - 2^{-2i})$.
2. Let $s \in \mathbb{Z}_{\geq 0}$. $\lambda(L_{\mathbb{Z}_2} \perp H; s) = (1 - 2^{-2s-2}) \cdot \lambda(L_{\mathbb{Z}_2}; s+1)$.
3. Let $s \in \mathbb{Z}_{\geq 0}$. For a unimodular lattice of even dimension of discriminant ε we have:

$$\lambda(L_{\mathbb{Z}_2}; s) = \prod_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} (1 - 2^{-2i-2s}) (1 - \left(\frac{\varepsilon}{2}\right) 2^{-\frac{m}{2}-s}).$$

Here $\left(\frac{\varepsilon}{2}\right) = (-1)^{\frac{\varepsilon^2-1}{8}}$ is the Kronecker symbol.

4. Let $s \in \mathbb{Z}_{\geq 0}$. For a lattice $L_{\mathbb{Z}_2}$ of the form $L' \perp \langle \varepsilon' \rangle$, where $L'_{\mathbb{Z}_2}$ is unimodular of discriminant ε , and $\varepsilon, \varepsilon' \in \mathbb{Z}_2^*$, we have:

$$\lambda(L_{\mathbb{Z}_2}; s) = |2|_2^{s+\frac{m-1}{2}} \prod_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} (1 - 2^{-2i-2s}).$$

5. Let $L_{\mathbb{Z}_2}$ be a lattice of the form $\langle \varepsilon_1, \varepsilon_2 \rangle, \varepsilon_i \in \mathbb{Z}_2^*$. We have

$$\lambda(L_{\mathbb{Z}_2}; 0) = \frac{1}{2}.$$

9. A simple illustration of the orbit equation

9.1. We give an easy example to illustrate the orbit equation:

Let $p \neq 2$ and $L_{\mathbb{Z}_p}$, be a unimodular lattice of *odd* dimension $m \geq 3$. We want to calculate

$$\mu_p(L_{\mathbb{Z}_p}, \langle \varepsilon p^a \rangle, L_{\mathbb{Z}_p}; s) = |p^a|_p^{-s + \frac{2-m}{2}} \beta_p(L_{\mathbb{Z}_p}, \langle \varepsilon p^a \rangle, L_{\mathbb{Z}_p}; s).$$

We first assume, a *odd*. On the one hand, by Yang's formula (Theorem 7.2), it is given by

$$|p^a|_p^{-s + \frac{2-m}{2}} \left(1 + (1 - p^{-1}) \sum_{k=1}^{\frac{a-1}{2}} p^{(2-m)k} X^{2k} - p^{(2-m)\frac{a+1}{2}-1} X^{a+1} \right),$$

where $X = p^{-s}$, as usual. On the other hand, we have $\frac{a+1}{2}$ orbits of vectors of length εp^a (Lemma A.7). The lattices $\alpha_i(\langle \varepsilon p^a \rangle)^\perp$ are of the form $\langle \varepsilon'' p^{2i-1} \rangle^\perp \perp L'_{\mathbb{Z}_p}$, where $L'_{\mathbb{Z}_p}$ is unimodular. We have by Theorem 8.1:

$$\lambda(\alpha_i(\langle \varepsilon p^a \rangle)^\perp; s) = |p^{2i-1}|_p^{s + \frac{m-2}{2}} \prod_{j=1}^{\frac{m-3}{2}} (1 - p^{-2j-2s})$$

and

$$\lambda(L_{\mathbb{Z}_p}; s) = \prod_{i=1}^{\frac{m-1}{2}} (1 - p^{-2i-2s}).$$

The orbit equation hence reduces to the following identity of rational functions in $X = p^{-s}$,

$$\begin{aligned} & \prod_{i=1}^{\frac{m-1}{2}} (1 - p^{-2i} X^2)^{-1} \\ & \cdot X^{-a} p^{a\frac{m-2}{2}} \left(1 + (1 - p^{-1}) \sum_{k=1}^{\frac{a-1}{2}} p^{(2-m)k} X^{2k} - p^{(2-m)\frac{a+1}{2}-1} X^{a+1} \right) \\ & = \sum_{i=1}^{\frac{a+1}{2}} X^{-(2i-1)} p^{(2i-1)\frac{m-2}{2}} \prod_{j=1}^{\frac{m-3}{2}} (1 - p^{-2j} X^2)^{-1}, \end{aligned}$$

which one can check easily in an elementary way.

If a is even, let $2\varepsilon'$ be the discriminant of $L_{\mathbb{Z}_p}$. By Yang's formula (Theorem 7.2), we get:

$$|p^a|_p^{-s + \frac{2-m}{2}} \left(1 + (1 - p^{-1}) \sum_{k=1}^{\frac{a}{2}} p^{(2-m)k} X^{2k} + \left((-1)^{\frac{m-1}{2}} \frac{\varepsilon \varepsilon'}{p} \right) p^{\frac{(2-m)(a+1)-1}{2}} X^{a+1} \right).$$

On the other hand, we have $\frac{a}{2} + 1$ orbits of vectors of length εp^a (Lemma A.7). The lattices $\alpha_i(< \varepsilon p^a >)^\perp$ are of the form $< \varepsilon'' p^{2i} >^\perp L'_{\mathbb{Z}_p}$, where $L'_{\mathbb{Z}_p}$ is unimodular. For $i = 0$, $\alpha_i(< \varepsilon p^a >)^\perp$ has discriminant $\varepsilon \varepsilon'$. We have by Theorem 8.1:

$$\lambda(\alpha_i(< \varepsilon p^a >)^\perp; s) = \prod_{j=1}^{\frac{m-3}{2}} (1 - p^{-2j-2s}) \begin{cases} 1 - \left(\frac{(-1)^{\frac{m-1}{2}} \varepsilon \varepsilon'}{p} \right) p^{-\frac{m-1}{2}-s} & i = 0, \\ |p^{2i}|^{s+\frac{m-2}{2}} & i > 0. \end{cases}$$

The orbit equation hence reduces to the following identity of rational functions in $X = p^{-s}$,

$$\begin{aligned} & \prod_{i=1}^{\frac{m-1}{2}} (1 - p^{-2i} X^2)^{-1} \\ & \cdot X^{-a} p^{a\frac{m-2}{2}} \left(1 + (1 - p^{-1}) \sum_{k=1}^{\frac{a}{2}} p^{(2-m)k} X^{2k} + \left(\frac{(-1)^{\frac{m-1}{2}} \varepsilon' \varepsilon}{p} \right) p^{\frac{(2-m)(a+1)-1}{2}} X^{a+1} \right) \\ & = \left(\frac{1}{1 - \left(\frac{(-1)^{\frac{m-1}{2}} \varepsilon \varepsilon'}{p} \right) p^{-\frac{m-1}{2}} X} + \sum_{k=1}^{\frac{a}{2}} X^{-2k} p^{k(m-2)} \right) \prod_{j=1}^{\frac{m-3}{2}} (1 - p^{-2j} X^2)^{-1}, \end{aligned}$$

which one can check easily in an elementary way.

10. A global orbit equation

Lemma 10.1. *Let $M_{\mathbb{Q}}, L_{\mathbb{Q}}$ be vector spaces of dimensions n, m respectively, where $n \leq m - 1$, with quadratic forms Q_M and Q_L . If $m \geq n + 3$, the product of the canonical measures on $I(M_{\mathbb{Q}_\nu}, L_{\mathbb{Q}_\nu})$ converges absolutely (in the sense of [34]) and yields the canonical measure on $I(M_{\mathbb{A}}, L_{\mathbb{A}})$.*

In the case $n = m$, the product of the canonical measures on $\mathrm{SO}(L_{\mathbb{Q}_\nu})$ converges absolutely and yields the canonical measure on $\mathrm{SO}(L_{\mathbb{A}})$, provided $m \geq 3$. It is the Tamagawa measure.

Proof. Follows directly from the explicit volume formulæ in the unimodular case (Theorem 8.1, 4., 5.) and standard facts about absolute convergence of the occurring infinite products. One just obtains the Tamagawa number in the second case because of the product formula $|x|_{\mathbb{A}} = 1$ for $x \in \mathbb{Q}^*$ for the adelic modulus, i.e. the discriminant factors cancel in the product. \square

10.2. Let $L_{\mathbb{Z}}$ be a lattice of dimension m with $Q_L \in \mathrm{Sym}^2(L_{\mathbb{Z}}^*)$ of signature $(m - 2, 2)$.

For 1-dimensional $M \cong \mathbb{Z}$, ($n = 1$), we have $Q \in \mathbb{Z}$ describing the quadratic form $x \mapsto Qx^2$ with associated ‘symmetric morphism’ $\gamma = 2Q$, and $\gamma_0 = 1$:

$$\begin{aligned} & \lim_{\alpha \rightarrow \infty} |\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^2 Q} E_Q(\Psi_\infty, \frac{m}{2} \Phi(\chi_\kappa); g_l(\alpha), s) \\ &= \lim_{\alpha \rightarrow \infty} |\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^2 Q} W_{Q,\infty}(\Psi_\infty, \frac{m}{2}(s), g_l(\alpha)) \prod_p W_{Q,p}(\Phi_p(\chi_\kappa; s), 1) \\ &= |\gamma|_\infty^s |2\gamma|_\infty^{\frac{1}{2}(s-s_0)} \mu_\infty(L_{\mathbb{Z}_p}, \langle Q \rangle, \kappa; s - s_0) \prod_p |\gamma|_p^s \mu_p(L_{\mathbb{Z}_p}, \langle Q \rangle, \kappa; s - s_0). \end{aligned}$$

(cf. 4.5 and 4.7, respectively).

This is the quantity, which can be related to Arakelov geometry, using the result of [17] or [4] (see section 11.6).

We therefore define

$$\mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) = \prod_{\nu} \mu_{\nu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s)$$

and

$$\tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) = |2d(M_{\mathbb{Z}})|_\infty^{-\frac{1}{2}s} \mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s).$$

With this definition we have in the 1-dimensional case:

$$\lim_{\alpha \rightarrow \infty} |\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^2 Q} E_Q(\Psi_\infty \Phi(\chi_\kappa); g_l(\alpha), s - s_0) = \tilde{\mu}(L_{\mathbb{Z}_p}, \langle Q \rangle, \kappa; s).$$

For $n > 1$ we do not now, if the global $\tilde{\mu}$ occurs as a limit in this fashion as well.

10.3. Let D be the discriminant of L . Assume, that $L_{\mathbb{Z}_p}$ splits n hyperbolic planes at all $p \neq 2$. Define

$$\lambda(L_{\mathbb{Z}}; s) := \prod_{\nu} \lambda_{\nu}(L_{\mathbb{Z}}; s)$$

and

$$\tilde{\lambda}(L; s) := |D|_\infty^{\frac{1}{2}s} \lambda(L_{\mathbb{Z}}; s),$$

where D is the discriminant of $L_{\mathbb{Z}}$. Sometimes we will use also $\tilde{\lambda}_p(\dots)$ for $|D|_p^{-\frac{1}{2}s} \lambda_p(\dots)$, and similarly $\tilde{\mu}_p(\dots) := |2d(M_{\mathbb{Z}})|_p^{\frac{1}{2}s} \mu_p(\dots)$.

Lemma 10.4. *If $L_{\mathbb{Z}}$ is of signature $(m-2, 2)$. Let $M_{\mathbb{Z}}$ be a positive definite lattice. Take a $\kappa \in (L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \otimes M_{\mathbb{Z}}^*$. Assume $m-n \geq 1$ and $I(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa \neq \emptyset$. $\mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s)$, $\lambda(L_{\mathbb{Z}}; s)$ have meromorphic continuations to the entire complex plane and are holomorphic and nonzero in a neighborhood of $s = 0$. Similarly for $\tilde{\mu}, \tilde{\lambda}$. They depend only on the genera of $L_{\mathbb{Z}}, M_{\mathbb{Z}}$.*

Proof. Follows directly from (8.1) and using standard facts about the occurring quadratic L -series. \square

In particular, for the meromorphic continuation of the Eisenstein series (14) above remains true also in exceptional cases.

We obviously get taking the product over all ν of equations (5.10) and (6.2), respectively:

$$\lambda^{-1}(L_{\mathbb{Z}}; s) \mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) = \sum_{\alpha \text{ SO}'(L_{\widehat{\mathbb{Z}}}) \subset \text{I}(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa} \lambda^{-1}(\alpha_{\mathbb{Z}}^{\perp}; s),$$

where $\alpha_{\mathbb{Z}}^{\perp}$ is a lattice such that $\alpha_{\mathbb{Z}}^{\perp} \otimes \mathbb{Z}_p \cong \alpha_{\mathbb{Z}_p}^{\perp}$ for all p . It can be realized in some class $L'_{\mathbb{Z}}$ in the genus of $L_{\mathbb{Z}}$ (2.5). λ depends only on its genus.

For the corresponding $\tilde{\mu}, \tilde{\lambda}$, this equation is not true anymore. However, we get at least, denoting by \mathbb{R}_N the reals modulo rational multiples of $\log(p)$, for $p \mid N$:

Theorem 10.5. *Assume $m \geq 3$, $m - n \geq 1$. Let D be the discriminant of $L_{\mathbb{Z}}$ and D' be the D -primary part of the discriminant of $M_{\mathbb{Z}}$.*

$$\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0) = \sum_{\alpha \text{ SO}'(L_{\widehat{\mathbb{Z}}}) \subset \text{I}(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa} \tilde{\lambda}^{-1}(\alpha_{\mathbb{Z}}^{\perp}; 0)$$

and

$$\frac{d}{ds} \left(\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) \right) \Big|_{s=0} \equiv \sum_{\alpha \text{ SO}'(L_{\widehat{\mathbb{Z}}}) \subset \text{I}(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa} \frac{d}{ds} \tilde{\lambda}^{-1}(\alpha_{\mathbb{Z}}^{\perp}; s) \Big|_{s=0}$$

in $\mathbb{R}_{2DD''}$, where D'' is the product of primes such that $p^2 \nmid D'$.

Proof. Follows by induction from the fact that for $p \nmid D$ and Q square-free at p there is only 1 orbit (A.7) in $\text{I}(< Q >, L)(\mathbb{Z}_p)$ and α^{\perp} has discriminant p at p . \square

We will need also a global version of Kitaoka's formula (5.3):

Theorem 10.6. *Assume $m \geq 3$, $m - n \geq 1$. Let D be the discriminant of $L_{\mathbb{Z}}$ and $M_{\mathbb{Z}} = M'_{\mathbb{Z}} \perp M''_{\mathbb{Z}}$. Let D' be the D -primary part of the discriminant of $M'_{\mathbb{Z}}$ (not $M_{\mathbb{Z}}$!). Let $\kappa \in (L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \otimes M_{\mathbb{Z}}^*$ with a corresponding decomposition $\kappa = \kappa' \oplus \kappa''$. We have*

$$\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0) = \sum_{\substack{\alpha \text{ SO}'(L_{\widehat{\mathbb{Z}}}) \subset \text{I}(M', L)(\mathbb{A}^{(\infty)}) \cap \kappa' \\ \kappa'' \cap \alpha_{\mathbb{A}^{(\infty)}}^{\perp} \otimes (M''_{\mathbb{A}^{(\infty)}})^* \neq \emptyset}} \tilde{\lambda}^{-1}(\alpha_{\mathbb{Z}}^{\perp}; 0) \tilde{\mu}(\alpha_{\mathbb{Z}}^{\perp}, M_{\widehat{\mathbb{Z}}}, \kappa''; 0)$$

and

$$\begin{aligned} & \frac{d}{ds} \left(\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) \right) \Big|_{s=0} \\ &= \sum_{\substack{\alpha \text{ SO}'(L_{\widehat{\mathbb{Z}}}) \subset \text{I}(M', L)(\mathbb{A}^{(\infty)}) \cap \kappa' \\ \kappa'' \cap \alpha_{\mathbb{A}^{(\infty)}}^{\perp} \otimes (M''_{\mathbb{A}^{(\infty)}})^* \neq \emptyset}} \frac{d}{ds} \left(\tilde{\lambda}^{-1}(\alpha_{\mathbb{Z}}^{\perp}; s) \tilde{\mu}(\alpha_{\mathbb{Z}}^{\perp}, M'', \kappa''; s) \right) \Big|_{s=0} \end{aligned}$$

in $\mathbb{R}_{2DD'}$. Here $\kappa'' \in L_{\widehat{\mathbb{Z}}}^*/L_{\widehat{\mathbb{Z}}}$ is considered as an element of $(\alpha_{\widehat{\mathbb{Z}}}^{\perp})^*/\alpha_{\widehat{\mathbb{Z}}}^{\perp} \otimes (M''_{\widehat{\mathbb{Z}}})^*$ via $\kappa'' \mapsto \kappa'' \cap \alpha_{\mathbb{A}^{(\infty)}}^{\perp} \otimes (M''_{\mathbb{A}^{(\infty)}})^*$.

Proof. Let first $s \in \mathbb{Z}_{\geq 0}$. For all $p \nmid 2DD'$, there is only one orbit (generated for $s = 0$ by α , say) in $I(M', L \oplus H^s)(\mathbb{Z}_p)$. Hence we have, using Kitaoka's formula 5.14:

$$\lambda_p^{-1}(L_{\mathbb{Z}_p}; s) \tilde{\mu}_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, L_{\mathbb{Z}_p}; s) = \lambda_p^{-1}(\alpha_{\mathbb{Z}_p}^\perp; s) \tilde{\mu}_p(\alpha_{\mathbb{Z}_p}^\perp, M_{\mathbb{Z}_p}'', \kappa''; s).$$

For all other p , we have the equation

$$\lambda_p^{-1}(L_{\mathbb{Z}_p}; 0) \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; 0) = \sum_{\substack{\alpha \text{ SO}'(L_{\mathbb{Z}_p}) \subset I(M', L)(\mathbb{Q}_p) \cap \kappa' \\ \kappa'' \cap \alpha_{\mathbb{Q}_p}^\perp \otimes (M_{\mathbb{Q}_p}'')^* \neq \emptyset}} \lambda_p^{-1}(\alpha_{\mathbb{Z}_p}^\perp; 0) \mu_p(\alpha_{\mathbb{Z}_p}^\perp, M'', \kappa''; 0).$$

and since the quantities $\tilde{\mu}_p$ are polynomials in p^{-s} in this case (cf. 4.5), the assertion is true. \square

11. Applications to Kudla's program

11.1. The investigations of recursive properties of representation densities in this paper were motivated by the program of Kudla (cf. [13, 14, 15, 16, 18, 23, 24, 25, 26, 27, 28]) to describe the relation of heights of special subvarieties of certain Shimura varieties to special derivatives of the Eisenstein series considered in section 4.3 of this paper. We briefly describe this here and report on related results obtained in the thesis of the author [11], which will be published in a forthcoming paper [10]. The object of study are the orthogonal Shimura varieties, which are defined over \mathbb{Q} ($m \geq 3$) and whose associated complex analytic orbifolds are given by

$$[\text{SO}(L_{\mathbb{Q}}) \backslash \mathbb{D}_{\mathbf{O}} \times (\text{SO}(L_{\mathbb{A}(\infty)})/K)],$$

where $L_{\mathbb{Q}}$ is a quadratic space of signature $(m-2, 2)$, $\mathbb{D}_{\mathbf{O}}$ is the associated (Hermitian) symmetric space and K is a compact open subgroup of $\text{SO}(L_{\mathbb{A}(\infty)})$. There is an injective intertwining map $h_{\mathbf{O}} : \mathbb{D}_{\mathbf{O}} \hookrightarrow \text{Hom}(\mathbb{S}, \text{SO}(L_{\mathbb{R}}))$ such that $\mathbf{O} := (P_{\mathbf{O}} := \text{SO}(L), \mathbb{D}_{\mathbf{O}}, h_{\mathbf{O}})$ is a Shimura datum [11, Definition 3.2.2]. If a lattice $L_{\mathbb{Z}}$ is chosen, \mathbf{O} is integral at all primes not dividing the discriminant D of $L_{\mathbb{Z}}$ (this means, that $P_{\mathbf{O}}$ extends to a reductive group scheme over $\mathbb{Z}_{(p)}$). In the first part of the thesis of the author [11, Main Theorems 4.2.2, 4.3.5], it is sketched that there is a *canonical* integral model $M_{\Delta}^K(\mathbf{O})$ of a toroidal compactification of the above variety defined over $\mathbb{Z}[1/2D]$ ($m \geq 3$) provided K is admissible, i.e. for all $p \nmid D$ of the form $\text{SO}(L_{\mathbb{Z}_p}) \times K^{(p)}$, where $K^{(p)}$ is a compact open subgroup of $\text{SO}(L_{\mathbb{A}(\infty, p)})^1$. The compactification depends on the additional datum of a rational polyhedral cone decomposition Δ . In [11, Main Theorem 4.5.2] it is also shown, that there exists a theory of Hermitian automorphic vector bundles on these models. For each SO-equivariant vector bundle \mathcal{E} on

$$M^{\vee}(\mathbf{O}) = \{ \langle v \rangle \in \mathbb{P}L \mid Q(v) = 0 \}$$

¹The proofs still rely on a technical assumption which remains open.

(the compact dual, defined over $\mathbb{Z}[1/2D]$, too) equipped with a $\mathrm{SO}(L_{\mathbb{R}})$ -invariant Hermitian metric on $\mathcal{E}_{\mathbb{C}}$, restricted to the image of the Borel embedding, there is an associated (canonically determined) Hermitian automorphic vector bundle $(\Xi^* \mathcal{E}, \Xi^* h)$ on $M(\frac{K}{\Delta} \mathbf{O})$ whose metric has logarithmic singularities along the boundary divisor of $M(\frac{K}{\Delta} \mathbf{O})(\mathbb{C})$. The associated analytic bundle, restricted to the uncompactified Shimura variety, can be canonically identified with

$$[\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathcal{E}_{\mathbb{C}}|_{\mathbb{D}_{\mathbf{O}}} \times (\mathrm{SO}(\mathbb{A}^{(\infty)})/K)],$$

equipped with the quotient of the given metric. The construction is functorial in morphisms of Shimura data, in particular, everything above commutes with inclusions of lattices. The results so far are conditional on a missing technical hypothesis [11, Conjecture 4.3.2]. In what follows, we let \mathcal{E} be the restriction of the tautological bundle on $\mathbb{P}L$ and h be a certain multiple [11, Definiton 11.3.1] of the metric $v \mapsto \langle v, \bar{v} \rangle$.

11.2. Let M be a positive definite space and $\varphi \in S(L_{\mathbb{A}^{(\infty)}} \otimes M_{\mathbb{A}^{(\infty)}}^*)$ a locally constant function with compact support. Provided that φ is K -invariant, this defines a **special cycle** $Z(L, M, \varphi; K)$ on these Shimura varieties, defined as

$$\sum_{K\alpha \subset \mathrm{I}(M, L)(\mathbb{A}^{(\infty)}) \cap \mathrm{supp}(\varphi)} \varphi(\alpha) [\mathrm{SO}(\alpha_{\mathbb{Q}}^{\perp}) \backslash \mathbb{D}_{\mathbf{O}(\alpha^{\perp})} \times \mathrm{SO}(\alpha_{\mathbb{A}^{(\infty)}}^{\perp}) / ({}^g K \cap \mathrm{SO}(\alpha_{\mathbb{A}^{(\infty)}}^{\perp}))].$$

Observe that the sum goes over finitely many orbits and that there is a natural embedding $\mathbb{D}_{\mathbf{O}(\alpha_{\mathbb{Z}}^{\perp})} \hookrightarrow \mathbb{D}_{\mathbf{O}}$ and an embedding $\mathrm{SO}(\alpha_{\mathbb{A}^{(\infty)}}^{\perp}) \rightarrow \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$, $h \mapsto hg^{-1}$, where g is the element defining $\alpha_{\mathbb{Z}}^{\perp}$ as an integral lattice (2.5). These cycles are naturally a weighted sum over sub-Shimura varieties of the same type. In the special case $K = \mathrm{SO}'(L_{\widehat{\mathbb{Z}}})$ and φ is the characteristic function of a coset in $(L_{\widehat{\mathbb{Z}}}^*/L_{\widehat{\mathbb{Z}}}) \otimes M_{\widehat{\mathbb{Z}}}^*$, these correspond precisely to the orbits in the global orbit equation 10.5.

11.3. We have the following relation between *values* at 0 of the orbit equation and volumes of special cycles. The equation says

$$4\lambda^{-1}(L_{\mathbb{Z}}; 0)\mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0) = 4 \sum_{\mathrm{SO}'(L_{\widehat{\mathbb{Z}}})\alpha \subset \mathrm{I}(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa} \lambda^{-1}(\alpha_{\mathbb{Z}}^{\perp}; 0).$$

It is well known that $4\lambda^{-1}(L_{\mathbb{Z}}, 0)$ equals the volume of $M(\mathrm{SO}'(L_{\widehat{\mathbb{Z}}})\mathbf{O})(\mathbb{C})$ w.r.t. the volume form $c_1(\Xi^* \mathcal{E}, \Xi^* h)^{m-2}$ — see e.g. [11, Theorem 11.5.2, i.]. Similarly $4\lambda^{-1}(\alpha_{\mathbb{Z}}^{\perp}; 0)$ is the volume of the sub-Shimura variety of the special cycle. Hence by definition of the special cycle (and canonicity of the construction Ξ^*), $\mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0)$ is equal to the degree of the special cycle (w.r.t. $\Xi^* \mathcal{E}, \Xi^* h$) divided by the volume of the surrounding Shimura variety. More generally, it is possible to define the special cycles also for degenerate quadratic forms on M and the equality with the special values of the Fourier coefficients of the Eisenstein series extends to those. Therefore the special value of the Eisenstein series is a generating function for these degrees. In a lot of cases, it is known

that in fact also their generating functions valued in cohomology or even Chow groups of the Shimura variety are modular. See the introduction to [11] and the references therein for more information on this.

11.4. According to the conjectures of Kudla and others, the first derivative of $\tilde{\mu}$, or more generally of the full Fourier coefficient of the Eisenstein series 4.3, should be related to *heights* of the special cycles. In [11, Theorem 11.5.2, ii., 11.5.5, 11.5.9], we proved a partial result for the nonsingular coefficients in this direction:

Theorem 11.5. *Assume $m - n > 1$. Under a general technical assumption (cf. 11.1), we have the following:*

	$M^{(\mathrm{SO}'(L_{\mathbb{Z}})\mathbf{O})}$	$Z(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa)$
geometric volume <i>w.r.t.</i> $c_1(\Xi^*\mathcal{E}_{\mathbb{C}}, \Xi^*h)$	$\prod_{\nu} \tilde{\lambda}_{\nu}^{-1}(L; 0)$	$\prod_{\nu} \tilde{\lambda}_{\nu}^{-1}(L; 0) \tilde{\mu}_{\nu}(L, M, \kappa; 0)$
arithmetic volume <i>w.r.t.</i> $\hat{c}_1(\Xi^*\mathcal{E}, \Xi^*h)$	$\frac{d}{ds} \prod_{\nu} \tilde{\lambda}_{\nu}^{-1}(L; s) \Big _{s=0}$ <i>up to $\mathbb{Q} \log(p)$ for $p^2 4d(L)$.</i>	$\frac{d}{ds} \prod_{\nu} \tilde{\lambda}_{\nu}(L; s) \tilde{\mu}_{\nu}(L, M, \kappa; s) \Big _{s=0}$ <i>up to $\mathbb{Q} \log(p)$ for $p 2d(L)$ and p such that $M_{\mathbb{Z}_p}^*/M_{\mathbb{Z}_p}$ is not cyclic.</i>

Here, in the first line, we repeated the well-known facts for the case of the geometric volume, mentioned above, for the sake of comparison.

Note that $\tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s)$ is the “holomorphic” part of the Fourier coefficient of the Eisenstein series 4.3. It is conjectured, that if one equips the special cycles with Kudla-Millson’s Greens functions [19, 20] and extends them (possibly in a nontrivial way, i.e. not by simply taking its Zariski closure) to the integral model, the generating series of their classes $[\hat{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa, y)]$ in appropriate arithmetic Chow groups should be modular forms, whose product with $\hat{c}_1(\Xi^*\mathcal{E}, \Xi^*h)^{m-n-1}$ should yield the *full* first derivative of the Eisenstein series. The Greens functions depend on $y \in (M \otimes M)_{\mathbb{R}}^s$ (the imaginary part of the argument of the Eisenstein series as classical modular form), too.

Whereas $\hat{c}_1(\Xi^*\mathcal{E}, \Xi^*h)$ is an object in an already developed [6, 7] extended Arakelov theory allowing additional boundary singularities of the occurring Greens function, this is not yet clear for these classes $[\hat{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa, y)]$. In any case, the additional ingredient would be to prove that the integral of the Kudla-Millson Greens function is given by the special derivative of the “nonholomorphic part” of the Fourier coefficient of the Eisenstein series in the following way:

$$\begin{aligned} & \int_{M^{(K)\mathbf{O}}(\mathbb{C})} \mathfrak{g}(Q, y) c_1(\Xi^* \mathcal{E}, \Xi^* h)^{m-n-1} \\ &= \text{vol}(Z(L_{\mathbb{Z}}, M_{\mathbb{Z}}^Q, \kappa; K)) \frac{d}{ds} \mu_{nh}(L_{\mathbb{Z}}, M_{\mathbb{Z}}^Q, \kappa; y, s) \Big|_{s=0}, \end{aligned}$$

where $\mu_{nh}(L_{\mathbb{Z}}, M_{\mathbb{Z}}^Q, \kappa; y, s)$, the “non-holomorphic” part of the Fourier coefficient of the Eisenstein series is determined by

$$\begin{aligned} & |\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^! Q \cdot \gamma_0^{-1}} E_Q(\Psi_{\infty} \Phi(\chi_{\kappa}); g_l(\alpha), s - s_0) \\ &= \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}^Q, \kappa; s) \mu_{nh}(L_{\mathbb{Z}}, M_{\mathbb{Z}}^Q, \kappa; \alpha \gamma_0^{-1}({}^t \alpha^{-1}), s). \end{aligned}$$

Its value at $s = 0$ is equal to 1, independently of α (10.2). See e.g. [27, Proposition 12.1] for a calculation of this kind for Shimura curves, which correspond to signature (1,2), Witt rank 0.

At primes, where the Zariski closure of the special cycle itself consists of canonical integral models of Shimura varieties, the height is equal (again by the compatibility of the Ξ^* construction with embeddings) to the (degree of the) highest power of $\widehat{c}_1(\Xi^* \mathcal{E}, \Xi^* h)$ computed on these various smaller models.

By Theorem 11.5, this “self-intersection number” or “arithmetic volume” of a Shimura variety associated with a lattice $L'_{\mathbb{Z}}$ and its discriminant kernel is given by the first derivative at $s = 0$ of

$$4\tilde{\lambda}^{-1}(L'_{\mathbb{Z}}; s)$$

up to multiples of $\log(p)$ dividing the discriminant of $L'_{\mathbb{Z}}$ and $p = 2$. For induced (but maybe not canonical in any reasonable sense) models one can extend this to be up to multiples of those $\log(p)$, where p^2 divides the discriminant. Obviously, the orbit equation relates this — applied to the individual sub-Shimura varieties in a special cycle — directly to the height (or geometric volume) of the special cycle. We emphasize, that the contribution of the RHS of the orbit equation *for very bad primes* is *not* conjectured to be related to the decomposition into sub-Shimura varieties. This is already wrong in the case of the modular curve (Shimura variety associated with the lattice considered in section 12) as we will illustrate in 11.8 below.

11.6. For $n = 1$, the global orbit equation at $s = 0$ and its derivative can be understood in terms of Borcherds products (and in the arithmetic case this is the key to the proof of 11.5!). This idea was first used in [3]. Recall that a Borcherds product is a meromorphic modular form on $M^{(K)\mathbf{O}}(\mathbb{C})$ having singularities precisely in the (codimension $n = 1$) special cycles. It is a multiplicative lift of an integral vector valued modular form f of weight $1 - \frac{m}{2}$ holomorphic in \mathbb{H} and meromorphic in the cusp $i\infty$ for the Weil representation of $\text{Sp}'_2(\mathbb{Z})$ restricted to $\mathbb{C}[L_{\mathbb{Z}}^*/L_{\mathbb{Z}}]$. Such an f has a Fourier expansion

$$f = \sum_{k \in \mathbb{Q} \gg -\infty} a_k q^k,$$

where only a_k 's with k having bounded denominator actually occur and $a_k \in \mathbb{Z}[L_{\widehat{\mathbb{Z}}}^*/L_{\widehat{\mathbb{Z}}}]$. The divisor of F (on the uncompactified Shimura variety) is given by $\sum_{k < 0} Z(L_{\mathbb{Z}}, < -k >, a_k)$ and its weight is equal to $a_0(0)$. The geometric case is now “explained” as follows: Using Serre duality one can show, that the nonpositive Fourier coefficients have to satisfy the equation

$$\sum_{k \in \mathbb{Q}_{\leq 0}} a_k b_{-k} = 0$$

for all *holomorphic* forms $\sum_{k \in \mathbb{Q}_{\geq 0}} b_k q^k$ of weight $\frac{m}{2}$ for the dual of the Weil representation restricted to $\mathbb{C}[L_{\widehat{\mathbb{Z}}}^*/L_{\widehat{\mathbb{Z}}}]$. The Eisenstein series (see 10.2), whose special value is holomorphic (assuming $m \geq 4$), yields the relation

$$c_0(0) = \sum_{k < 0} \mu(L_{\mathbb{Z}}, < -k >, a_k; 0).$$

If we assume for the moment that there is only one nonzero $a_k, k < 0$ equal to the characteristic function of κ , we may confirm that $\mu(L_{\mathbb{Z}}, < -k >, \kappa; 0)$ gives the relative degree of $Z(L_{\mathbb{Z}}, < -k >, \kappa)$ w.r.t. $\Xi^* \mathcal{E}, \Xi^* h$ (modulo investigations at the boundary).

The arithmetic case is “explained” as follows: We assume again for simplicity that there is only one nonzero $a_k, k < 0$ equal to the characteristic function of κ . In [11, Theorem 11.3.11] we showed (using Borchers’ product expansion and an abstract integral q -expansion principle for automorphic vector bundles) that for good p these Borchers forms are actually rational sections of the bundle $\Xi^* \mathcal{E}$ on any smooth compactified integral model $M(\frac{K}{\Delta} \mathbf{O})$, having no connected (=irreducible) fibre above p in their divisor. Their divisor hence is supported on the Zariski closures of the special divisors and possibly on the boundary. The derivative of the orbit equation now reads as follows:

$$\begin{aligned} & 4(\tilde{\lambda}^{-1})'(L_{\mathbb{Z}}; 0) \tilde{\mu}(L_{\mathbb{Z}}, < -k >, \kappa; 0) + 4\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) \tilde{\mu}'(L_{\mathbb{Z}}, < -k >, \kappa; 0) \\ &= 4 \sum_{SO'(L_{\widehat{\mathbb{Z}}}) \alpha \subset I(< Q >, L)(\mathbb{A}^{(\infty)}) \cap \kappa} (\tilde{\lambda}^{-1})'(\alpha_{\widehat{\mathbb{Z}}}^{\perp}; 0). \end{aligned}$$

$4(\tilde{\lambda}^{-1})'(\alpha_{\widehat{\mathbb{Z}}}^{\perp}; 0)$ is interpreted as the arithmetic volume/height of the sub-Shimura variety in the special divisor corresponding to the orbit of α . Hence the value of the equation is the height of the special divisor (cf. Theorem 11.5).

$$4(\tilde{\lambda}^{-1})'(L_{\mathbb{Z}}; 0) \tilde{\mu}(L_{\mathbb{Z}}, < -k >, \kappa; 0)$$

is the arithmetic volume of the surrounding Shimura variety multiplied by the weight of F , and finally, according to the computation of Kudla and Bruinier/Kühn (compare 10.2 with [17, Theorem 2.12 (ii)] or use [4, Proposition 3.2, Theorem 4.11] and 7.3)

$$-4\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) \tilde{\mu}'(L_{\mathbb{Z}}, < -k >, \kappa; 0)$$

is equal to the integral of the logarithm of the Hermitian norm of F over $M^{(K)\mathbf{O}}(\mathbb{C})$ (taking into account, that $4\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0)$ is the volume of $M^{(K)\mathbf{O}}(\mathbb{C})$). The equation hence (again modulo — considerably difficult — investigations at the boundary) expresses the following standard relation in Arakelov geometry:

$$\begin{aligned} \widehat{\deg}(\widehat{\operatorname{div}}(F) \cdot \widehat{c}_1(\Xi^* \mathcal{E}, \Xi^* h)^{m-2}) &= \widehat{\deg}(\widehat{c}_1((\Xi^* \mathcal{E}, \Xi^* h)|_{\overline{Z(L_{\mathbb{Z}}, < -k >, \kappa)}})^{m-2}) \\ &+ \int_{M^{(K)\mathbf{O}}(\mathbb{C})} \log h(F) c_1(\Xi^* \mathcal{E}, \Xi^* h)^{m-2}. \end{aligned}$$

Observe $\widehat{\operatorname{div}}(F) = \mu(L_{\mathbb{Z}}, < -k >, \kappa; 0) \widehat{c}_1(\Xi^* \mathcal{E}, \Xi^* h)$, so the LHS is equal to $\mu(L_{\mathbb{Z}}, < -k >, \kappa; 0) \widehat{\deg}(\widehat{c}_1(\Xi^* \mathcal{E}, \Xi^* h)^{m-1})$. All values are understood in \mathbb{R} modulo contribution from $\log(p)$ for p dividing the discriminant of $L_{\mathbb{Z}}$ (not $M_{\mathbb{Z}} = < -k >!$) and for $p = 2$.

11.7. Consider a decomposition $M_{\mathbb{Z}} = M'_{\mathbb{Z}} \perp M''_{\mathbb{Z}}$. The derivative of the global Kitaoka formula 10.6 at $s = 0$, which reads

$$4\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) = 4 \sum_{\substack{SO'(L_{\widehat{\mathbb{Z}}}) \alpha \subset I(M', L)(\mathbb{A}^{(\infty)}) \cap \kappa' \\ \kappa'' \cap \alpha_{\mathbb{A}^{(\infty)}}^{\perp} \otimes (M''_{\mathbb{A}^{(\infty)}})^* \neq \emptyset}} \tilde{\lambda}^{-1}(\alpha_{\mathbb{Z}}^{\perp}; s) \tilde{\mu}^{-1}(\alpha_{\mathbb{Z}}^{\perp}, M'', \kappa''; s)$$

just reflects (in view of Theorem 11.5) the following elementary equality of cycles:

$$Z(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa) = \sum_{\substack{SO'(L_{\widehat{\mathbb{Z}}}) \alpha \subset I(M', L)(\mathbb{A}^{(\infty)}) \cap \kappa' \\ \kappa'' \cap \alpha_{\mathbb{A}^{(\infty)}}^{\perp} \otimes (M''_{\mathbb{A}^{(\infty)}})^* \neq \emptyset}} Z(\alpha_{\mathbb{Z}}^{\perp}, M'', \kappa'').$$

11.8. In the case of the modular curve, i.e. if L is the lattice discussed in section 12, we have explicitly by Kronecker's limit formula:

$$2E(S; s+1) = \operatorname{vol}(M^{(K')\mathbf{O}}(S^{\perp})) + \operatorname{ht}(M^{(K')\mathbf{O}}(S^{\perp}))s + O(s^2)$$

(where the K' denote the respective discriminant kernels $SO'(S_{\widehat{\mathbb{Z}}}^{\perp})$) and in the sum over all $SO'(L_{\widehat{\mathbb{Z}}})$ orbits:

$$2E(< q >, \kappa; s+1) = \operatorname{vol}(Z(L_{\mathbb{Z}}, < q >, \kappa)) + \operatorname{ht}(Z(L_{\mathbb{Z}}, < q >, \kappa))s + O(s^2)$$

Therefore, Theorem 11.5 in this case (its geometric part is then roughly the classical class number formula) follows from Theorem 12.5, 5. Observe, that the latter identity is an identity of functions in s , not only an identity of the first 2 Taylor coefficients! Therefore one might ask, if there is any arithmetic or K -theoretic “explanation” of the equality of the higher terms in the Taylor expansion, too. Observe furthermore that the derivative of $E(S; s+1)$ at $s = 0$ is *not* related in any simple way to the expansion of $4\lambda^{-1}(S_{\widehat{\mathbb{Z}}}^{\perp}; s)$ at $s = 0$, if S^{\perp} has nonsquarefree discriminant (cf. also 12.8). Therefore one cannot expect a simple direct relation of the derivative of the orbit equation in this naive form to Arakelov geometry which is true without any restriction.

11.9. According to Theorem 11.5, the value, resp. first derivative of $4\tilde{\lambda}^{-1}(L_{\mathbb{Z}})$ encode the geometric, resp. arithmetic volume of the orthogonal Shimura varieties associated with $L_{\mathbb{Z}}$ and its discriminant kernel, in the second case up to rational multiples of $\log(p)$ with either $p = 2$ or a prime with $p^2 | D(L_{\mathbb{Z}})$. We will illustrate the computation of geometric and arithmetic volume of several Shimura varieties associated with lattices with square-free discriminant (outside 2) in the rest of this section. We use the explicit computation (8.1, 8.2) and bring them to a form involving derivatives of L -series at negative integers as it is common in the literature.

Example 11.10 (Heegner points). *Let $L_{\mathbb{Z}}$ be a two dimensional negative definite lattice with square-free discriminant $D > 0$ (in particular $2 \nmid D$ and $-D$ is automatically fundamental). We have*

$$\begin{aligned} \tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) &= L(\chi_{-D}, 0) \\ &\quad + L(\chi_{-D}, 0) \cdot \left(-\frac{L'(\chi_{-D}, 0)}{L(\chi_{-D}, 0)} + \frac{1}{2}C - \frac{1}{2}\log(D) + \frac{1}{2}\log(2) \right) s \\ &\quad + O(s^2) \end{aligned}$$

Example 11.11 (Modular curves). *Let $L_{\mathbb{Z}}$ be a three dimensional lattice with form $x_1x_2 - \varepsilon x_3^2$, ε square free, $2 \nmid \varepsilon$. The discriminant D is 2ε .*

$$\begin{aligned} \tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) &= -\frac{1}{2}\zeta(-1) \prod_{p|\varepsilon} (p+1) - \frac{1}{2}\zeta(-1) \cdot \\ &\quad \cdot \prod_{p|\varepsilon} (p+1) \left(-2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} \sum_{p|\varepsilon} \frac{p-1}{p+1} \log(p) - 1 + C + \frac{1}{2}\log(2) \right) s \\ &\quad + O(s^2) \end{aligned}$$

Example 11.12 (Shimura curves). *Let $L_{\mathbb{Z}}$ be a three dimensional lattice again with discriminant $D = 2\varepsilon$, $2 \nmid \varepsilon$, ε square-free, and assume that the form is anisotropic at all $p|\varepsilon$.*

$$\begin{aligned} \tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) &= -\frac{1}{2}\zeta(-1) \prod_{p|\varepsilon} (p-1) - \frac{1}{2}\zeta(-1) \cdot \\ &\quad \cdot \prod_{p|\varepsilon} (p-1) \left(-2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} \sum_{p|\varepsilon} \frac{p+1}{p-1} \log(p) - 1 + C + \frac{1}{2}\log(2) \right) s \\ &\quad + O(s^2) \end{aligned}$$

Example 11.13 (Hilbert modular surfaces). *Let $L_{\mathbb{Z}}$ a four dimensional lattice, being an orthogonal direct sum of a two dimensional indefinite of discriminant*

$D < 0$, square free and a hyperbolic plane. The we have:

$$\begin{aligned}\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) &= \frac{1}{4}\zeta(-1)L(\chi_{-D}, -1) \\ &\quad + \frac{1}{4}\zeta(-1)L(\chi_{-D}, -1) \cdot \\ &\quad \cdot \frac{1}{4} \left(-2 \frac{\zeta'(-1)}{\zeta(-1)} - \frac{L'(\chi_{-D}, -1)}{L(\chi_{-D}, -1)} - \frac{3}{2} - \frac{1}{2} \log(D) + \frac{3}{2}C + \frac{1}{2} \log(2) \right) s \\ &\quad + O(s^2)\end{aligned}$$

Example 11.14 (Siegel threefolds). *Let $L_{\mathbb{Z}}$ be a five dimensional lattice, which is the orthogonal sum of the negative of a three dimensional as in example (11.12) and a hyperbolic plane. Let $D = 2\varepsilon > 0$ be the discriminant of $L_{\mathbb{Z}}$*

$$\begin{aligned}\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) &= -\frac{1}{4}\zeta(-1)\zeta(-3) \prod_{p|D} (p^2 - 1) - \frac{1}{4}\zeta(-1)\zeta(-3) \prod_{p|D} (p^2 - 1) \cdot \\ &\quad \cdot \left(-2 \frac{\zeta'(-1)}{\zeta(-1)} - 2 \frac{\zeta'(-3)}{\zeta(-3)} + \frac{1}{2} \sum_{p|D} \frac{p^2 + 1}{p^2 - 1} \log(p) - \frac{17}{6} + 2C + \frac{1}{2} \log(2) \right) s \\ &\quad + O(s^2)\end{aligned}$$

Example 11.15 (A 10-dimensional Shimura variety). *Especially simple is the situation for a Shimura variety of orthogonal type associated with an unimodular lattice (this has good reduction everywhere, except possibly $p = 2$). Let for example $L_{\mathbb{Z}}$ be the orthogonal direct sum of a positive definite E_8 -lattice and 2 hyperbolic planes. Here we get:*

$$\begin{aligned}\lambda^{-1}(L_{\mathbb{Z}}; s) &= \frac{1}{16}\zeta(-1)\zeta(-3)\zeta^2(-5)\zeta(-7)\zeta(-9) \\ &\quad + \frac{1}{16}\zeta(-1)\zeta(-3)\zeta^2(-5)\zeta(-7)\zeta(-9) \cdot \left(-2 \frac{\zeta'(-1)}{\zeta(-1)} - 2 \frac{\zeta'(-3)}{\zeta(-3)} \right. \\ &\quad \left. - 3 \frac{\zeta'(-5)}{\zeta(-5)} - 2 \frac{\zeta'(-7)}{\zeta(-7)} - 2 \frac{\zeta'(-9)}{\zeta(-9)} - \frac{14717}{1260} + \frac{11}{2}C \right) s \\ &\quad + O(s^2)\end{aligned}$$

(and here this coincides with $\tilde{\lambda}^{-1}$).

12. The example $ac - b^2$

12.1. In this section, we investigate the special case of the following \mathbb{Z} -lattice:

$$L_{\mathbb{Z}} = \{A \in M_2(\mathbb{Z}) \mid {}^t A = A\}$$

with quadratic form $Q : L_{\mathbb{Z}} \rightarrow \mathbb{Z}$, $A \mapsto \det(A)$. It has discriminant 2 and signature (1,2). The group scheme GL_2 acts on L by conjugation. This identifies PGL_2 with the special orthogonal group scheme $\mathrm{SO}(L)$ of the lattice. The discriminant kernel $\mathrm{SO}'(L_{\mathbb{Z}})$ is equal to the special orthogonal group $\mathrm{SO}(L_{\mathbb{Z}})$.

12.2. Via \perp , maximal negative definite sublattices $N \cap L_{\mathbb{Z}}$ correspond bijectively to vectors $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ of positive length $ac - b^2 > 0$ with $a < 0, c < 0, 2|a, 2|c$ and $(\frac{a}{2}, b, \frac{c}{2}) = 1$. They correspond also to 2 complex vectors of length 0

$$Z_{\tau^{\pm}} = \begin{pmatrix} 1 & \tau^{\pm} \\ \tau^{\pm} & (\tau^{\pm})^2 \end{pmatrix},$$

too, with $\tau^+ \in \mathbb{H}, \tau^- = \overline{\tau^+}$, by virtue of

$$a(\tau^{\pm})^2 - 2b\tau^{\pm} + c = 0, \text{ i.e. } \tau^{\pm} = \frac{b \pm i\sqrt{\det(S)}}{a}$$

$$N_{\tau} = Z_{\tau^+} \oplus Z_{\tau^-} = \mathbb{R} \begin{pmatrix} \frac{a}{2} & 0 \\ 0 & -\frac{c}{2} \end{pmatrix} + \mathbb{R} \begin{pmatrix} b & \frac{c}{2} \\ \frac{c}{2} & 0 \end{pmatrix}$$

Remark 12.3. The lattice $N_{\tau} \cap L_{\mathbb{Z}} = S_{\mathbb{Z}}^{\perp}$ is not necessarily equivalent to the lattice \mathbb{Z}^2 with form $x \mapsto \frac{1}{2}xSx$. However, the discriminant of $N_{\tau} \cap L_{\mathbb{Z}}$ is equal to $\det(S)$ if S satisfies the conditions of 12.2 (the generators of N given above span a lattice of discriminant $\frac{c^2}{4}\det(S)$ and of index $\frac{c}{2}$ in a primitive one).

12.4. Let

$$E'(h, s) = \sum_{g \in P(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{Q})} \Psi(gh)(s)$$

be the standard Eisenstein series of weight 0. Here $\Psi = \prod_{\nu} \Psi_{\nu}$, the Ψ_{ν} s are the standard sections:

$$\Psi_{\nu} \left(\begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} u(\beta)k \right) = |\alpha_1|_{\nu}^s |\alpha_2|_{\nu}^{-s},$$

however with different normalization than in (4.2). We have

$$E'(g_{\tau}, s) = \frac{1}{2} \sum_{g \in \Gamma_{\infty} \backslash \mathrm{SL}_2(\mathbb{Z})} |\Im(g \circ \tau)|^s = \frac{1}{2} \sum_{\substack{(m, n) \in \mathbb{Z} \\ \gcd(m, n) = 1}} \frac{|y|^s}{|m\tau + n|^{2s}}$$

for $g_{x+yi} = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ and $\Gamma_{\infty} = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Z} \}$.

The Eisenstein series (in this normalization) satisfies the functional equation [5, Theorem 1.6.1]:

$$Z(2s)E'(g, s) = Z(2-2s)E'(g, 1-s).$$

where $Z(s) = \zeta(s)\Gamma(\frac{1}{2}s)\pi^{-\frac{1}{2}s}$ is the normalized Riemann zeta function, satisfying $Z(s) = Z(1-s)$. We normalize the Eisenstein series as follows:

$$E(z; s) := \frac{Z(2s)}{Z(s)} E'(z; s).$$

For each integer q and $\kappa \in L_{\widehat{\mathbb{Z}}}^*/L_{\widehat{\mathbb{Z}}}$, we consider the decomposition of $I(< q >, L)(\mathbb{A}^{(\infty)}) \cap \kappa$ into orbits $SO'(L_{\widehat{\mathbb{Z}}})S$ (w.l.o.g. $S \in L_{\mathbb{Q}}$). We are interested in the ‘partial traces’:

$$E(S, s) := \sum_{g_f \in SO(S^\perp) \setminus SO(S_{\mathbb{A}^{(\infty)}}^\perp) / SO'(S_{\widehat{\mathbb{Z}}}^\perp)} \frac{1}{\# SO'(S_{\widehat{\mathbb{Z}}}^\perp)} E(g_\tau g_f, s)$$

(here $SO(S_{\mathbb{Q}}^\perp)$ is a torus in $SO(L_{\mathbb{Q}})$ and $\tau \in \mathbb{H}$ is such that $Z_\tau \perp S$, observe $E(g_\tau g_f, s) = E(g_{\overline{\tau}} g_f, s)$) and in the ‘complete traces’:

$$E(< q >, \kappa; s) := \sum_{SO'(L_{\widehat{\mathbb{Z}}})S \subset I(< q >, L)(\mathbb{A}^{(\infty)}) \cap \kappa} E(S; s).$$

Since $L_{\mathbb{Z}}$ has class number 1, w.l.o.g. $S \in I(< q >, L)(\mathbb{Q})$ and the latter can also be written as follows:

$$E(< q >, \kappa; s) = \sum_{SO(L_{\mathbb{Z}})S \subset I(< q >, L)(\mathbb{Q}) \cap \kappa} \frac{1}{\# SO'(S_{\mathbb{Z}}^\perp)} E(g_\tau; s).$$

The aim of this section is to prove the following (well-known)

Theorem 12.5.

$$\begin{aligned} E(S, s) &= \prod_{\nu} E_{\nu}(S, s), \\ E(< q >, \kappa; s) &= \prod_{\nu} E_{\nu}(< q >, \kappa; s), \end{aligned}$$

where $E_{\infty}(< q >, \kappa; s) = E_{\infty}(S, s)$ is independent of $S \in I(< q >, L)(\mathbb{Q}) \cap \kappa$ and $E_p(< q >, \kappa; s)$ is defined by the orbit equation:

$$E_p(< q >, \kappa; s) := \sum_{SO'(L_{\mathbb{Z}_p})S_p \subset I(< q >, L)(\mathbb{Q}_p) \cap \kappa_p} E_p(S_p, s). \quad (14)$$

We have explicitly $E_p(S, s) = |D(S_{\mathbb{Z}_p}^\perp)|_p^{-\frac{s}{2}} \frac{\zeta_p(S_{\mathbb{Z}_p}^\perp; s)}{\zeta_p(s)}$, where $\zeta_p(S_{\mathbb{Z}_p}^\perp, s)$ is the normalized zeta function (cf. Definition 7.5) of the lattice $S_{\mathbb{Z}_p}^\perp$ and $\zeta_p(s) = \frac{1}{1-p^{-s}}$.

The relation to representation densities is determined by

Theorem 12.6.

1. $E_{\infty}(< q >; s) = E_{\infty}(S; s) = 2\lambda_{\infty}^{-1}(S_{\mathbb{Z}}^\perp; s-1)$
2. If $p \neq 2$, $E_p(< q >; s) = \tilde{\lambda}_p^{-1}(L_{\mathbb{Z}_p}; s-1) \tilde{\mu}_p(L_{\mathbb{Z}_p}, < q >; s-1)$
3. If $p \neq 2$, $\nu_p(q) \leq 1$, we have:

$$E_p(S; s) = E_p(< q >; s) = \tilde{\lambda}_p^{-1}(S_{\mathbb{Z}}^\perp; s-1)$$

4. If $p \neq 2$, the $E_p(S; s)$ satisfy the following functional equation:

$$\frac{E_p(S; 1-s)}{L_p(\chi_{-q}; 1-s)} = \frac{E_p(S; s)}{L_p(\chi_{-q}; s)}.$$

5. In the product:

$$E(< q >, \kappa; s) = 2\tilde{\lambda}^{-1}(L_{\mathbb{Z}_p}; s-1)\tilde{\mu}(L_{\mathbb{Z}_p}, < q >, \kappa; s-1)$$

and if q is square-free:

$$E(S; s) = E(< q >; s) = 2\tilde{\lambda}^{-1}(S_{\mathbb{Z}}^{\perp}; s-1)$$

in both cases maybe up to a rational function in 2^{-s} .

Recall that $\tilde{\mu}(L_{\mathbb{Z}_p}, < q >, \kappa; s)$ itself is the ‘holomorphic part’ of a Fourier coefficient (of index q) of another, metaplectic Eisenstein series 10.2.

Proof of Theorem 12.5. If $E(< q >, s)$ is defined via equation (14), the second product expansion follows immediately from the first and the definition. Hence it is enough to show

$$E(S, s) = \frac{\Gamma(\frac{s}{2} + \frac{1}{2})}{\pi^{\frac{s}{2} + \frac{1}{2}}} |\det(S)|_{\infty}^{\frac{s}{2}} \frac{\zeta(S_{\mathbb{Z}}^{\perp}; s)}{\zeta(s)}.$$

for S normalized as in 12.2.

First we have

$$\begin{aligned} E'(S; s) &= \sum_{h_f \in \mathrm{SO}(S_{\mathbb{Q}}^{\perp}) \backslash \mathrm{SO}(S_{\mathbb{A}(\infty)}^{\perp}) / \mathrm{SO}'(S_{\mathbb{Z}}^{\perp})} \frac{1}{\#\mathrm{SO}'(S_{\mathbb{Z}}^{\perp})} \sum_{g \in P(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{Q})} \Psi(s; gg_{\tau}h_f) \\ &= \sum_{h \in \mathrm{SO}(S_{\mathbb{A}(\infty)}^{\perp}) / \mathrm{SO}'(S_{\mathbb{Z}}^{\perp})} \Psi(s; hg_{\tau}), \end{aligned}$$

where we used $\mathrm{PGL}_2(\mathbb{Q}) = P(\mathbb{Q}) \mathrm{SO}(S_{\mathbb{Q}}^{\perp})$ in the last step. Considering the right invariance of $\Psi(s)$ under K , this may be rewritten as:

$$E'(S; s) = \Psi_{\infty}(s; g_{\tau}) \prod_p \frac{\int_{\mathbb{G}_m(\mathbb{Q}_p) \backslash T(\mathbb{Q}_p)} \Psi_p(s; x) \mu(x)}{\mathrm{vol}_{\mu}(K \cap T(\mathbb{A}^{(\infty)}))},$$

where we denoted the preimage of $\mathrm{SO}(S^{\perp})$ in GL by T and μ is any translation invariant measure on $\mathbb{G}_m(\mathbb{Q}_p) \backslash T(\mathbb{Q}_p)$.

Lemma 12.7 shows that it is convenient to parameterize the torus T as follows (recall $a \neq 0$):

$$\begin{aligned} \iota : \mathbb{Q}_p^2 - \{0\} &\rightarrow T(\mathbb{Q}_p) \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} x + 2\frac{b}{a}y & \frac{c}{a}y \\ -y & x \end{pmatrix} \end{aligned}$$

We have $\det(\iota(v)) = \frac{Q(v)}{\frac{a}{2}}$, where $Q(v) = \frac{a}{2}x^2 + bxz + \frac{c}{2}y^2$. A translation invariant measure on $T(\mathbb{A}^{(\infty)})$ is given by $\det(\iota(v))^{-1} dv$, where dv is the standard measure on \mathbb{Q}_p^2 . We choose the measure μ , quotient of this by a measure giving \mathbb{Z}_p^* volume 1.

This yields (using the lemma):

$$\int_{\mathbb{Z}_p^2} \left| \frac{Q(v)}{\frac{a}{2}} \right|_p^{s-1} dv = \sum_{i=0.. \infty} p^{-2is} \int_{\mathbb{G}_m(\mathbb{Q}_p) \backslash T(\mathbb{Q}_p)} \Psi_p(s; x) \mu$$

and therefore

$$E'(S; s) = \frac{2^{-s} |\det(S)|_{\infty}^{\frac{1}{2}s}}{\zeta(2s)} \prod_p \frac{\int_{\mathbb{Z}_p^2} |Q(v)|_p^{s-1} dv}{\left| \frac{a}{2} \right|_p^{-1} \text{vol}\{x, y \mid \frac{c}{a}y, x, x + 2\frac{b}{a}y, y \in \mathbb{Z}_p, x^2 + 2\frac{b}{a}xy + \frac{c}{a}y^2 \in \mathbb{Z}_p^*\}}.$$

Substituting $\frac{a}{2}y$ for y and then $x - \frac{b}{2}y$ for x in the volume computation in the denominator, we get:

$$\left| \frac{a}{2} \right|_p^{-1} \text{vol}\{\dots\} = \text{vol}\{x, y \in \mathbb{Z}_p^2 \mid x^2 + \frac{ac - b^2}{4}y^2 \in \mathbb{Z}_p^*\},$$

if b is even or $p \neq 2$, otherwise substitute in addition $x + \frac{1}{2}$ for x and get

$$\left| \frac{a}{2} \right|_p^{-1} \text{vol}\{\dots\} = \text{vol}\{x, y \in \mathbb{Z}_p^2 \mid x^2 + xy + \frac{ac - b^2 + 1}{4}y^2 \in \mathbb{Z}_p^*\}.$$

Hence in any case:

$$E'(S; s) = 2^{-s} \frac{|\det(S)|_{\infty}^{\frac{1}{2}s}}{\zeta(2s)} \prod_p \zeta_p(S_{\mathbb{Z}_p}^{\perp}, s),$$

where $\zeta_p(S_{\mathbb{Z}_p}^{\perp}, s)$ is the normalized local zeta function of the lattice $S_{\mathbb{Z}_p}^{\perp}$ (see 7.5). Note, that it depends only on the discriminant for 2 dimensional lattices.

Hence

$$E(S; s) = \frac{Z(2s)}{Z(s)} E'(S; s) = 2^{-s} \frac{\Gamma(s) \pi^{\frac{s}{2}}}{\pi^s \Gamma(\frac{s}{2})} |\det(S)|^{\frac{s}{2}} \frac{\zeta(S_{\mathbb{Z}}^{\perp}; s)}{\zeta(s)}.$$

Applying the doubling formula for the Γ -function, we get indeed

$$E(S; s) = \frac{\Gamma(\frac{s}{2} + \frac{1}{2})}{\pi^{\frac{s}{2} + \frac{1}{2}}} |\det(S)|^{\frac{s}{2}} \frac{\zeta(S_{\mathbb{Z}}^{\perp}; s)}{\zeta(s)}.$$

□

Lemma 12.7. *We have $\Psi_{\infty}(s; g_{\tau}) = \left| \frac{\sqrt{\det(S)}}{a} \right|_{\infty}^s$ and $\Psi_p(s; h) = \left| \frac{\det(h)}{\text{ggT}(h_{12}, h_{22})^2} \right|_p^s$.*

Proof. The first assertion follows from the evaluation of the relation of g_τ and S , given above. For the second assertion consider the case $\nu_p(h_{12}) > \nu_p(h_{22})$, hence $ggT(h_{12}, h_{22}) = h_{22}$.

$$\begin{pmatrix} 1 & -h_{21}\det(h)^{-1}h_{22} \\ & 1 \end{pmatrix} \begin{pmatrix} \det(h)^{-1}h_{22} & \\ & h_{22}^{-1} \end{pmatrix} \begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix} = \begin{pmatrix} 1 & \\ \frac{h_{12}}{h_{22}} & 1 \end{pmatrix}$$

where the matrix on the right hand side is integral. Analogously for the case $\nu_p(h_{12}) < \nu_p(h_{22})$. \square

Proof of Theorem 12.6. 1. came out in the proof of Theorem 12.5.

2. Let $p \neq 2$, $S \in L_{\mathbb{Z}_p}$ and $Q(S) = q$. Denote $l := \nu_p(q)$. We have to show

$$\sum_{\mathrm{SO}'(L_{\mathbb{Z}_p})S \subset \mathrm{I}(\langle q \rangle, L)(\mathbb{Q}_p) \cap \kappa} |d(S_{\mathbb{Z}_p}^\perp)|_p^{-\frac{s}{2}} \frac{\zeta_p(S_{\mathbb{Z}_p}^\perp; s)}{\zeta_p(s)} = \tilde{\lambda}_p^{-1}(L_{\mathbb{Z}_p}; s-1) \tilde{\mu}_p(L_{\mathbb{Z}_p}, \langle q \rangle, \kappa; s-1)$$

Using the orbit equation, this can be rewritten as:

$$\begin{aligned} & \sum_{\mathrm{SO}'(L_{\mathbb{Z}_p})S \subset \mathrm{I}(\langle q \rangle, L)(\mathbb{Q}_p) \cap \kappa} |d(S_{\mathbb{Z}_p}^\perp)|_p^{-\frac{s}{2}} \frac{\zeta_p(S_{\mathbb{Z}_p}^\perp; s)}{\zeta_p(s)} \\ &= |q|_p^{\frac{1}{2}(s-1)} \sum_{\mathrm{SO}'(L_{\mathbb{Z}_p})S \subset \mathrm{I}(\langle q \rangle, L)(\mathbb{Q}_p) \cap \kappa} \lambda_p^{-1}(S_{\mathbb{Z}_p}^\perp; s-1) \end{aligned}$$

In case $l = 0$, we get:

$$\begin{aligned} \zeta_p(S_{\mathbb{Z}_p}^\perp; s) &= \frac{1}{(1-X)(1 - \left(\frac{-\varepsilon'}{p}\right)X)} \\ \lambda_p^{-1}(S_{\mathbb{Z}_p}^\perp; s-1) &= \frac{1}{1 - \left(\frac{-\varepsilon'}{p}\right)X} \end{aligned}$$

and we have just 1 orbit, hence the equation is true.

In case l odd, we have $\frac{l+1}{2}$ orbits (Lemma A.7) and get (Theorem 8.1 and 7.6):

$$\begin{aligned} \sum_S |d(S_{\mathbb{Z}_p}^\perp)|_p^{-\frac{s}{2}} \zeta_p(S_{\mathbb{Z}_p}^\perp; s) &= \sum_{k=0}^{\frac{l-1}{2}} X^{-\frac{1}{2}-k} \frac{1 - (pX^2)^{k+1} - X + X(pX^2)^k}{(1-X)(1-pX^2)}, \\ \sum_S \lambda_p^{-1}(S_{\mathbb{Z}_p}^\perp; s-1) &= \sum_{k=0}^{\frac{l-1}{2}} (pX)^{-2k-1} p^{k+\frac{1}{2}} \end{aligned}$$

Hence the equation reduces to

$$\sum_{k=0}^{\frac{l-1}{2}} X^{-\frac{1}{2}-k} \frac{1 - (pX^2)^{k+1} - X + X(pX^2)^k}{1 - pX^2} = (pX)^{\frac{l}{2}} \sum_{k=0}^{\frac{l-1}{2}} (pX)^{-2k-1} p^{k+\frac{1}{2}}$$

which may be checked in an elementary way.

In case $l \geq 2$ even, we have $\frac{l}{2} + 1$ orbits (Lemma A.7) and get (Theorem 8.1 and 7.6):

$$\begin{aligned} \sum_S |d(S_{\mathbb{Z}_p}^\perp)|_p^{-\frac{s}{2}} \zeta_p(S_{\mathbb{Z}_p}^\perp; s) &= \frac{1}{(1-X)(1 - \left(\frac{-\varepsilon'}{p}\right)X)} \\ &+ \sum_{k=1}^{\frac{l}{2}} X^{-k} \left(\frac{p^k X^{2k} - p^{k-1} X^{2k-1}}{(1-X)(1 - \left(\frac{-\varepsilon}{p}\right)X)} \right. \\ &\quad \left. + \frac{1 - (pX^2)^k - X + X(pX^2)^{k-1}}{(1-X)(1 - pX^2)} \right) \\ \sum_S \lambda^{-1}(S^\perp; s-1) &= \frac{1}{1 - \left(\frac{-\varepsilon'}{p}\right)X} + \sum_{k=1}^{\frac{l}{2}} (pX)^{-2k} p^k \end{aligned}$$

Hence the equation reduces to

$$\begin{aligned} \frac{1}{1 - \left(\frac{-\varepsilon'}{p}\right)X} + \sum_{k=1}^{\frac{l}{2}} X^{-k} \left(\frac{p^k X^{2k} - p^{k-1} X^{2k-1}}{1 - \left(\frac{-\varepsilon}{p}\right)X} + \frac{1 - (pX^2)^k - X + X(pX^2)^{k-1}}{1 - pX^2} \right) \\ = (pX)^{\frac{l}{2}} \left(\frac{1}{1 - \left(\frac{-\varepsilon'}{p}\right)X} + \sum_{k=1}^{\frac{l}{2}} (pX)^{-2k} p^k \right) \end{aligned}$$

which may again be checked in an elementary way.

3. is seen by a comparison of the formulæ obtained for $l = 0$ and $l = 1$ in the proof of 2. above with the explicit formulæ for λ in Theorem 8.1.

4. is true in the product by means of the functional equation of the Eisenstein series. The local statement follows because the functions p^{-s} are algebraically independent for different primes. Alternatively one can check this equation from the explicit formulæ obtained in the proof of 2.

5. is obtained out of 2. and 3. by taking the product over all ν . \square

Remark 12.8. *Theorem 12.6, 2. involves an identity between sums over orbits in $I(< q >, L_{\mathbb{Z}_p})$. These identities are striking from an elementary point of view, since the individual terms do not match. The terms coming from the traces over the Eisenstein series are directly related to the heights of individual sub-Shimura varieties of the modular curve (cf. also 11.8). Hence, in this case, the orbit equation we derived in this work cannot be related to Arakelov geometry (including information from p), if there is more than one orbit (at p).*

A. Some basic facts about quadratic forms

Lemma A.1. *Let $X \subset L_{\mathbb{Z}_p}^*$ be a subset and $\text{Stab}(X) \subset \text{SO}'(L_{\mathbb{Z}_p})$ the pointwise stabilizer. Assume that $X_{\mathbb{Q}_p}$ is non-degenerate. Then we have*

$$\text{Stab}(X) = \text{SO}'(X_{\mathbb{Z}_p}^\perp),$$

where $X_{\mathbb{Z}_p}^\perp = \{v \in L_{\mathbb{Z}_p} \mid \langle v, w \rangle = 0 \ \forall w \in X\}$.

Proof. Let $\beta \in \text{SO}'(L_{\mathbb{Z}_p})$, i. e. $\beta v - v \in L_{\mathbb{Z}_p}$ for all $v \in L_{\mathbb{Z}_p}^*$. If we have $\beta w = w$ for all $w \in X$, one can consider β via restriction as an element of $\text{SO}(X_{\mathbb{Z}_p}^\perp)$. If moreover $w^\perp \in (X_{\mathbb{Z}_p}^\perp)^*$, we find a $v \in L_{\mathbb{Z}_p}^*$ satisfying $v = w + w^\perp$ with some $w \in X$ (since $X_{\mathbb{Z}_p}^\perp$ is a primitive sublattice). This yields $\beta w^\perp - w^\perp = \beta v - v \in L_{\mathbb{Z}_p}$. Hence $\text{Stab}(X) \subseteq \text{SO}'(X_{\mathbb{Z}_p}^\perp)$.

On the other hand, let $\beta \in \text{SO}'(X_{\mathbb{Z}_p}^\perp)$. It can be extended uniquely to an element $\beta \in \text{SO}(L_{\mathbb{Q}_p})$, fixing $X_{\mathbb{Q}_p}$ pointwise. We claim that this extension lies in fact in $\text{SO}(L_{\mathbb{Z}_p})$. For, let $v \in L_{\mathbb{Z}_p}$ be given and write $v = w^\perp + w$. Then we have $w^\perp \in (X_{\mathbb{Z}_p}^\perp)^*$. hence $\beta v - v = \beta w^\perp - w^\perp \in L_{\mathbb{Z}_p}$.

Now suppose $v \in L_{\mathbb{Z}_p}^*$. Then we have still $w^\perp \in (X_{\mathbb{Z}_p}^\perp)^*$ because $\langle v, w^{\perp'} \rangle = \langle w^\perp, w^{\perp'} \rangle \in \mathbb{Z}_p$ for all $w^{\perp'} \in X_{\mathbb{Z}_p}^\perp$. Hence $\beta v - v \in L_{\mathbb{Z}_p}$ as well, which means $\beta \in \text{SO}'(L_{\mathbb{Z}_p})$ \square

Lemma A.2. *Let $M_{\mathbb{Z}_p}$ an unimodular sublattice of $L_{\mathbb{Z}_p}$. Then we have*

$$L_{\mathbb{Z}_p} = M_{\mathbb{Z}_p} \perp M_{\mathbb{Z}_p}^\perp.$$

Proof. Follows from [12, Prop. 5.2.2]. \square

Lemma A.3. [12, Theorem 5.2.2] *If $L_{\mathbb{Z}_p}$ is unimodular, we have*

$$L_{\mathbb{Z}_p} \simeq H_{\mathbb{Z}_p}^r \perp L_{\mathbb{Z}_p}^0,$$

with $L_{\mathbb{Z}_p}^0$ anisotropic. Here $H_{\mathbb{Z}_p}$ is an hyperbolic plane.

The following is well-known:

Lemma A.4. *Let R be a discrete valuation ring with $|2| = 1$ or a field and L_R a lattice with quadratic form $Q_L \in \text{Sym}^2(L_K^*)$, where K is the quotient field of R . There is a basis e_1, \dots, e_m of L_R , with respect to which the Q_L is given by*

$$Q_L : x \mapsto \sum_i \varepsilon_i p^{\nu_i} x_i^2,$$

where $\varepsilon_i \in \mathbb{Z}_{(p)}^*$, $\nu_i \in \mathbb{Z}$ and $\nu_1 \leq \dots \leq \nu_m$.

Lemma A.5. *Let $p \neq 2$. Assume $L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}$ is cyclic. Then*

$$\mathrm{SO}(L_{\mathbb{Z}_p})/\mathrm{SO}'(L_{\mathbb{Z}_p}) = \begin{cases} 1 & \text{if } \nu = 0 \text{ or } m = 1, \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise,} \end{cases}$$

where $p^\nu = |D(L_{\mathbb{Z}_p})|^{-1}$ is the order of $L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}$.

Proof. In the representation given by Lemma A.4 ν_m is equal to ν and all other ν_i vanish. Hence $v := p^{-\nu}e_m$ is a generator of $L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}$ and $Q(v) = \varepsilon_m p^{-\nu}$. Let v' be its image under an arbitrary isometry. v' has a representation

$$v' = \sum_{i < m} \alpha_i e_i + p^{-\nu} \alpha_m e_m \quad \alpha_i \in \mathbb{Z}_p$$

and

$$Q(v') = \sum_{i < m} \varepsilon_i \alpha_i^2 + \varepsilon_m p^{-\nu} \alpha_m^2 = \varepsilon_m p^{-\nu}.$$

From this it follows

$$\alpha_m^2 \equiv 1 \pmod{p^\nu},$$

hence ($p \neq 2$)

$$\alpha \equiv \pm 1 \pmod{p^\nu}.$$

The occurring sign defines a character of the orthogonal group. An element is in its kernel, precisely if it is in the discriminant kernel. Moreover, if $m > 1$ there are elements in SO , which yield sign -1 , for example composition of reflection along e_m and any $e_i, i < m$. \square

Lemma A.6. *Assume $p \neq 2$ and let*

$$L_{\mathbb{Z}_p} = M_{\mathbb{Z}_p} \perp M'_{\mathbb{Z}_p} = N_{\mathbb{Z}_p} \perp N'_{\mathbb{Z}_p},$$

with $\beta : M_{\mathbb{Z}_p} \cong N_{\mathbb{Z}_p}$. Then we have

$$M'_{\mathbb{Z}_p} \cong N'_{\mathbb{Z}_p}.$$

In particular, there exists an isometry $\alpha \in \mathrm{SO}(L_{\mathbb{Z}_p})$ with $\alpha(M_{\mathbb{Z}_p}) = N_{\mathbb{Z}_p}$. If $M_{\mathbb{Z}_p}$ is unimodular, we may choose $\alpha \in \mathrm{SO}'(L_{\mathbb{Z}_p})$. If $M'_{\mathbb{Z}_p}$ has a vector of unit length, we may assume in addition, that $\alpha|_{M'_{\mathbb{Z}_p}} = \beta$.

Proof. The first part of the assertion is shown in [12, Corollary 5.3.1]. It remains to see that we may choose the isometry in the discriminant kernel, if $M_{\mathbb{Z}_p}$ is unimodular: For this we proceed by induction on the dimension on $M_{\mathbb{Z}_p}$. If $M_{\mathbb{Z}_p}$ is one dimensional, let v be a generating vector of unit length and v' its image under β . One of the vectors $v + v'$ or $v - v'$ has unit length, call it w . The reflection along w lies obviously in $\mathrm{O}'(L_{\mathbb{Z}_p})$ and interchanges $\langle v \rangle$ and $\langle v' \rangle$. By composition with the reflection along v' , we may assume, that it lies in $\mathrm{SO}'(L_{\mathbb{Z}_p})$.

Assume now $\dim(M_{\mathbb{Z}_p}) > 1$. Let v be a vector of unit length in $M_{\mathbb{Z}_p}$ and v' its image. We have

$$L_{\mathbb{Z}_p} = \langle v \rangle \perp v^\perp = \langle v' \rangle \perp v'^\perp,$$

(Lemma A.2) and

$$v^\perp L_{\mathbb{Z}_p} = v^\perp M_{\mathbb{Z}_p} \perp M'_{\mathbb{Z}_p} \quad v'^\perp L_{\mathbb{Z}_p} = v'^\perp N_{\mathbb{Z}_p} \perp N'_{\mathbb{Z}_p},$$

and we have an isometry (case above) in the discriminant kernel, which maps $\langle v \rangle$ to $\langle v' \rangle$, and hence v^\perp to v'^\perp . $v^\perp M_{\mathbb{Z}_p}$ and $v'^\perp N_{\mathbb{Z}_p}$ are now isomorphic (again by the case above) hence (induction hypothesis), there is an isometry in $\mathrm{SO}'(v'^\perp)$ mapping the image of $v^\perp M_{\mathbb{Z}_p}$ to $v'^\perp N_{\mathbb{Z}_p}$. It lifts to $\mathrm{SO}'(L_{\mathbb{Z}_p})$. Composition with the first isometry gives the induction step. The proof shows, that we may arrange $\alpha|_{M_{\mathbb{Z}_p}} = \beta$ if there is a reflection in $\mathrm{SO}'(M'_{\mathbb{Z}_p})$. \square

Lemma A.7. *Let $L_{\mathbb{Z}_p}$ ($\dim(L) \geq 3$) be a unimodular lattice, $p \neq 2$, and $q \in \mathbb{Z}_p \setminus \{0\}$.*

Then $\mathrm{SO}(L_{\mathbb{Z}_p})$ acts transitively on $\{\alpha \in \mathrm{I}(\langle q \rangle, L_{\mathbb{Z}_p}) \mid \mathrm{im}(\alpha) \text{ is saturated}\}$. In particular, it acts transitively on $\mathrm{I}(\langle q \rangle, L_{\mathbb{Z}_p})$ with $|q|_p = \frac{1}{p}$.

Proof. Take any v with $Q_L(v) = q$. Diagonalize the form (Lemma A.4) and take the reflection v' of v at any basis vector e_i with the property that $v_i \in \mathbb{Z}_p^*$ (this must exist, since otherwise the vector would not be primitive). We have $p \nmid \langle v, v' \rangle$. Therefore the form on $\mathbb{Z}_p v \oplus \mathbb{Z}_p v'$ is unimodular, hence $\mathbb{Z}_p v \oplus \mathbb{Z}_p v'$ is primitive and a direct summand by Lemma A.2. It is necessarily a hyperbolic plane, since modulo p it represents zero. We have shown that any primitive vector in $\mathrm{I}(\langle q \rangle, L)$ lies in a hyperbolic plane. Now use Lemma A.6 and the fact, that $\mathrm{O}(H_{\mathbb{Z}_p})$ (not SO !) acts transitively on primitive vectors of length q on $H_{\mathbb{Z}_p}$. \square

We see that for $p \neq 2, j > 0$, and an unimodular lattice $L_{\mathbb{Z}_{(p)}}$, there are precisely $\lfloor \frac{j}{2} \rfloor + 1$ orbits of vectors of length p^j , indexed according to their ‘saturatedness’.

References

- [1] M. Abramowitz, I.A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, volume 55, National Bureau of Standards Applied Mathematics Series, 1964.
- [2] R.E. Borcherds, Automorphic forms with singularities on Grassmannians, Invent. Math. 132 (1998) 491–562.
- [3] J.H. Bruinier, J.I. Burgos Gil, U. Kühn, Borcherds products and arithmetic intersection theory on Hilbert modular surfaces, Duke Math. J. 139 (2007) 1–88.
- [4] J.H. Bruinier, U. Kühn, Integrals of automorphic Green’s functions associated to Heegner divisors, Int. Math. Res. Not. 31 (2003) 1687–1729.

- [5] D. Bump, Automorphic forms and representations, volume 55 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1997.
- [6] J.I. Burgos Gil, J. Kramer, U. Kühn, Arithmetic characteristic classes of automorphic vector bundles, *Doc. Math.* 10 (2005) 619–716 (electronic).
- [7] J.I. Burgos Gil, J. Kramer, U. Kühn, Cohomological arithmetic Chow rings, *J. Inst. Math. Jussieu* 6 (2007) 1–172.
- [8] B. Gross, W. Kohnen, D. Zagier, Heegner points and derivatives of L -series. II, *Math. Ann.* 278 (1987) 497–562.
- [9] B.H. Gross, D.B. Zagier, Heegner points and derivatives of L -series, *Invent. Math.* 84 (1986) 225–320.
- [10] F. Hörmann, The arithmetic and geometric volume of Shimura varieties of orthogonal type, 2010. To be submitted.
- [11] F. Hörmann, The arithmetic volume of Shimura varieties of orthogonal type, Ph.D. thesis, Humboldt-Universität zu Berlin, 2010.
- [12] Y. Kitaoka, Arithmetic of quadratic forms, volume 106 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 1993.
- [13] S.S. Kudla, Algebraic cycles on Shimura varieties of orthogonal type, *Duke Math. J.* 86 (1997) 39–78.
- [14] S.S. Kudla, Central derivatives of Eisenstein series and height pairings, *Ann. of Math. (2)* 146 (1997) 545–646.
- [15] S.S. Kudla, Derivatives of Eisenstein series and arithmetic geometry, in: *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, Higher Ed. Press, Beijing, 2002, pp. 173–183.
- [16] S.S. Kudla, Modular forms and arithmetic geometry, in: *Current developments in mathematics*, Int. Press, Somerville, MA, 2002, pp. 135–179.
- [17] S.S. Kudla, Integrals of Borchers forms, *Compositio Math.* 137 (2003) 293–349.
- [18] S.S. Kudla, Special cycles and derivatives of Eisenstein series, in: *Heegner points and Rankin L -series*, volume 49 of *Math. Sci. Res. Inst. Publ.*, Cambridge Univ. Press, Cambridge, 2004, pp. 243–270.
- [19] S.S. Kudla, J.J. Millson, The theta correspondence and harmonic forms. I, *Math. Ann.* 274 (1986) 353–378.
- [20] S.S. Kudla, J.J. Millson, The theta correspondence and harmonic forms. II, *Math. Ann.* 277 (1987) 267–314.

- [21] S.S. Kudla, S. Rallis, On the Weil-Siegel formula, *J. Reine Angew. Math.* 387 (1988) 1–68.
- [22] S.S. Kudla, S. Rallis, A regularized Siegel-Weil formula: the first term identity, *Ann. of Math. (2)* 140 (1994) 1–80.
- [23] S.S. Kudla, M. Rapoport, Arithmetic Hirzebruch-Zagier cycles, *J. Reine Angew. Math.* 515 (1999) 155–244.
- [24] S.S. Kudla, M. Rapoport, Cycles on Siegel threefolds and derivatives of Eisenstein series, *Ann. Sci. École Norm. Sup. (4)* 33 (2000) 695–756.
- [25] S.S. Kudla, M. Rapoport, Height pairings on Shimura curves and p -adic uniformization, *Invent. Math.* 142 (2000) 153–223.
- [26] S.S. Kudla, M. Rapoport, T. Yang, On the derivative of an Eisenstein series of weight one, *Internat. Math. Res. Notices* 7 (1999) 347–385.
- [27] S.S. Kudla, M. Rapoport, T. Yang, Derivatives of Eisenstein series and Faltings heights, *Compos. Math.* 140 (2004) 887–951.
- [28] S.S. Kudla, M. Rapoport, T. Yang, Modular forms and special cycles on Shimura curves, volume 161 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, 2006.
- [29] V. Platonov, A. Rapinchuk, Algebraic groups and number theory, volume 139 of *Pure and Applied Mathematics*, Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.
- [30] S. Rallis, L -Functions and the oscillator representation, volume 1245 of *Lecture Notes in Math.*, Springer, 1983.
- [31] G. Shimura, Confluent hypergeometric functions on tube domains, *Math. Ann.* 260 (1982) 269–302.
- [32] A. Weil, Sur certains groupes d’opérateurs unitaires, *Acta. Math.* 111 (1964) 143–211.
- [33] A. Weil, Sur la formule de Siegel dans la théorie des groupes classiques, *Acta. Math.* 113 (1965) 1–87.
- [34] A. Weil, Adeles and algebraic groups, volume 23 of *Progress in Mathematics*, Birkhäuser, 1982. With appendices by M. Demazure and T. Ono.
- [35] T. Yang, An explicit formula for local densities of quadratic forms, *J. Number Theory* 72 (1998) 309–356.